Equations with many friends

An introduction to integrable systems

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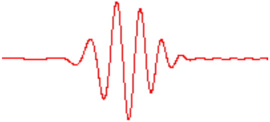
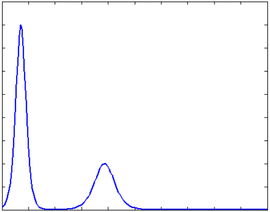
Linear vs nonlinear differential equations

Linear equations are boring . . . .

Nonlinear equations are difficult . . . .



Integrable systems are nonlinear equations that pretend to be linear



Click images for animations!

Hamiltonian Systems

Hamilton function

$$H : \mathbb{R}^{2N} \cong T^*Q \to \mathbb{R} : (q,p) \mapsto H(q,p)$$

determines dynamics:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Geometric interpretation:

- Phase space T^*Q with canonical symplectic 2-form ω
- flow along vector field X_H determined by $\iota_{X_H}\omega = \mathrm{d}H$
- the flows consists of symplectic maps and preserves H.

Poisson Brackets

Poisson bracket of two functionals on T^*Q:

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

Dynamics of a Hamiltonian system:

$$\dot{q}_i = \{q_i, H\}, \qquad \dot{p}_i = \{p_i, H\}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} f(q, p) = \{f(q, p), H\}$$

Properties:

$$\begin{aligned} \text{anti-symmetry:} \quad & \{f, g\} = -\{g, f\} \\ \text{bilinearity:} \quad & \{f, g + \lambda h\} = \{f, g\} + \lambda \{f, h\} \\ \text{Leibniz property:} \quad & \{f, gh\} = \{f, g\} h + g \{f, h\} \\ \text{Jacobi identity:} \quad & \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \end{aligned}$$

Liouville-Arnold integrability

What if $H(p, q) = p_i$?

$$\dot{p}_j = -\frac{\partial H}{\partial q_i} = 0 \qquad \text{and} \qquad \dot{q}_j = \frac{\partial H}{\partial p_i} = \delta_{i,j}.$$

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A Hamiltonian system with Hamilton function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville-Arnold integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \ldots H_N$ such that $\{H_i, H_j\} = 0$.

This implies that

- the flows commute.
- each H_i is a conserved quantity for all flows.
- the dynamics is confined to a leaf of the foliation $\{H_i = \text{const}\}$.
- There exists a symplectic change of variables $(p, q) \mapsto (H, T)$. Then

$$H_i(H, T) = H_i$$

Liouville-Arnold integrable systems evolve linearly in these variables!

Example: Kepler problem

Physical Hamiltonian:

$$H_2 = \frac{1}{2}|p|^2 - \frac{1}{|q|}$$

$$\Rightarrow \begin{cases} \dot{q} = p \\ \dot{p} = -\frac{q}{|q|^3} \end{cases}$$

Additional Hamiltonian:

$$H_1 = p^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} q$$

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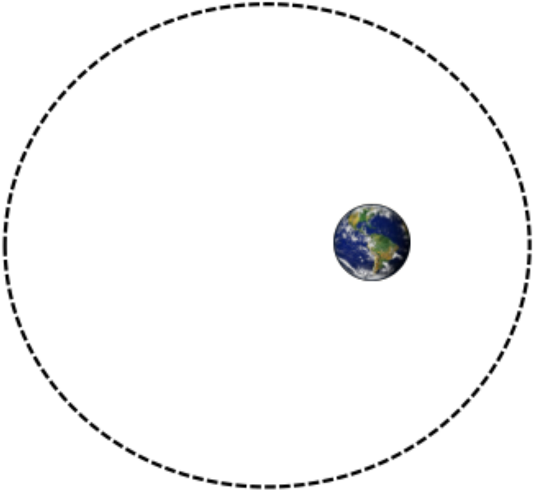

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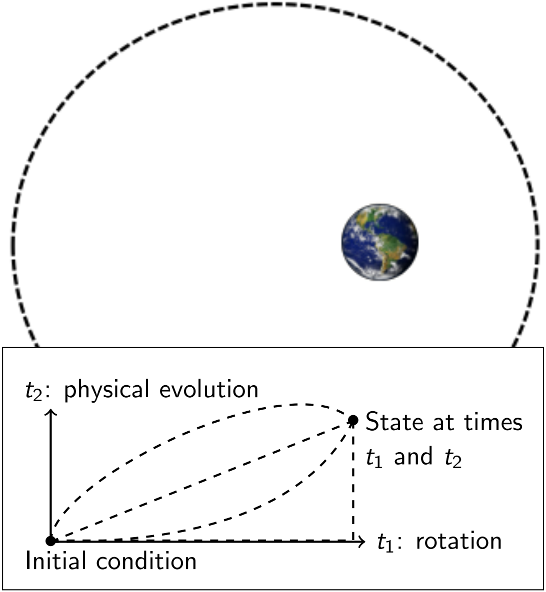


Lax Pairs

A Lax Pair consists of two matrices (or operators) L and P,

- depending on the dynamical variables,
- acting on some auxiliary space,
- such that

$$\frac{\mathrm{d}}{\mathrm{d}t} L = [P, L] \qquad (:= PL - LP) \qquad (*)$$

is equivalent to the equations of motion.

Equation $(*)$ represents the compatibility of

- the eigenvalue problem $L(t)\phi(t) = \lambda\phi(t)$,
- and the linear equation $\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = P(t)\phi(t)$.

We get conserved quantities for free:

$$\mathrm{tr}(L^k) = \text{const}$$

Example: harmonic oscillator

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & -\frac{1}{2}\omega \\ \frac{1}{2}\omega & 0 \end{pmatrix}$$

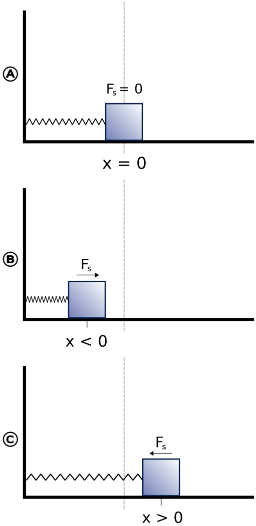
$$\frac{\mathrm{d}}{\mathrm{d}t} L = [P, L] = PL - LP$$

$$\Rightarrow \begin{pmatrix} \dot{p} & \omega \dot{q} \\ \omega \dot{q} & -\dot{p} \end{pmatrix} = \begin{pmatrix} -\omega^2 q & \omega p \\ \omega p & \omega^2 q \end{pmatrix}$$

Hence $\quad \dot{p} = -\omega^2 q \quad$ and $\quad \dot{q} = p$

Conserved quantities:

$$\begin{aligned} \mathrm{tr}(L) &= 0, \\ \mathrm{tr}(L^2) &= 2p^2 + 2\omega^2 q^2 = 4H, \\ \mathrm{tr}(L^3) &= 0, \\ \mathrm{tr}(L^4) &= 2(p^2 + \omega^2 q^2)^2 = 8H^2, \\ &\dots \end{aligned}$$



Example: Korteweg-de Vries (KdV) hierarchy

Lax Pair description:

- $\dfrac{\mathrm{d}}{\mathrm{d}t_k} L = [P_k, L]$ with $L = \partial^2 + v$.
- Only makes sense if $[P_k, L]$ is real-valued.
- This is the case for

$$P_3 = \partial^3 + \frac{3}{2} v \partial + \frac{3}{4} v_x,$$

$$P_5 = \partial^5 + \frac{5}{2} v \partial^3 + \frac{15}{4} v_x \partial^2 + \left(\frac{25}{8} v_{xx} + \frac{15}{8} v^2 \right) \partial + \left(\frac{15}{16} v_{xxx} + \frac{15}{8} v v_x \right)$$

$$\dots$$

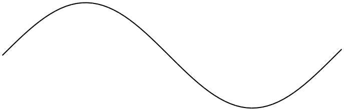
Equations: $v_{t_3} = v_{xxx} + 6 v v_x$ $\quad\rightarrow$ shallow water waves (click for movie)

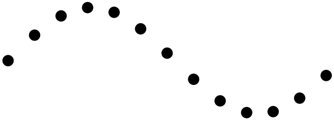
$$v_{t_5} = v_{xxxxx} + 20 v_x v_{xx} + 10 v v_{xxx} + 30 v^2 v_x$$

$$\dots$$

The problem of integrable discretization

- Many notions of integrability have a discrete counterpart
  $\rightarrow$ Integrable difference equations.
- Numerical discretizations almost always destroy integrability.
- What is the link between the continuous and discrete worlds?





Quad equations

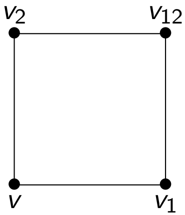
$$Q(v, v_1, v_2, v_{12}, \alpha_1, \alpha_2) = 0 \quad \text{on } \mathbb{Z}^2$$

Subscripts of v denote lattice shifts,

α_1, α_2 are parameters.

Invariant under symmetries of the square.

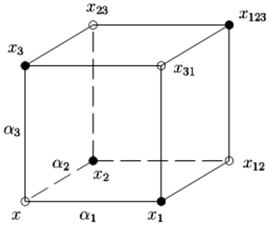
Affine in each of v, v_1, v_2, v_{12}.



Integrability for systems quad equations: multi-dimensional consistency of

$$Q(v, v_i, v_j, v_{ij}, \alpha_i, \alpha_j) = 0,$$

i.e. given v, v_1, v_2 and v_3, the three ways of calculating v_{123} give the same result.



Example: discrete KdV equation

$$(v - v_{12})(v_1 - v_2) - \alpha_1 + \alpha_2 = 0$$

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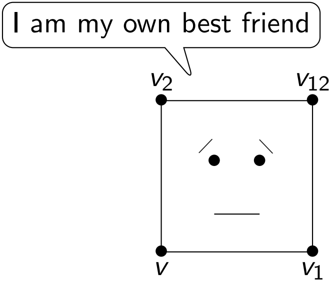

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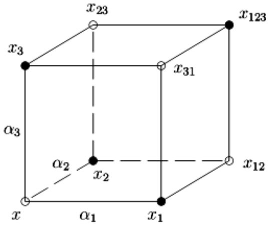

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Integrable continuum limits

Consider the continuous world $\mathbb{R}^{n+1}(x, t_1, \ldots, t_n)$

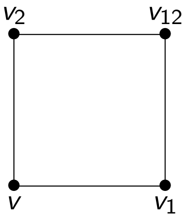
Identify each lattice shift with some shift in $\mathbb{R}^{n+1}$, depending on α_1, α_2.

Then we can write

$$Q(v, v_1, v_2, v_{12}, \alpha_1, \alpha_2) = 0$$

as a power series:

$$\sum_{i,j \in \mathbb{N}} F_{i,j}\, \alpha_1^i \alpha_2^j = 0.$$



Example: for the discrete KdV equation, we find that $F_{0,j} = 0$ ($j = 2, 4, \ldots$) is the KdV hierarchy.

(Other $F_{i,j}$ are trivial modulo KdV equations)

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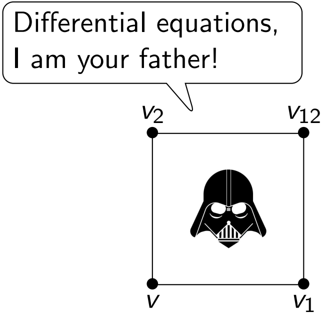

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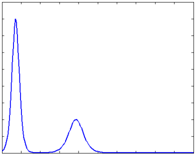
Discrete integrable systems seem to be the fundamental objects.

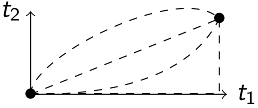
Discrete theory is well-developed, e.g. classification.

Can we translate such results to the continuous world?

Summary



Integrable systems pretend to be linear.



They can do so because they have many friends.

A single integrable difference equation can generate a full family of integrable differential equations.

Suggested reading

O. Babelon, D. Bernard, M. Talon. Introduction to classical integrable systems. Cambridge University Press, 2003.

Yu. Suris. The problem of integrable discretization: Hamiltonian approach. Birkhäuser, 2012.

F. Nijhoff. Discrete Systems and Integrability. Lecture Notes, University of Leeds.