A variational structure for integrable hierarchies

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(Optimistic) Table of Contents

[1] Suris, V. On the Lagrangian structure of integrable hierarchies. In Advances in Discrete Differential Geometry, Springer, 2016.

[2] Petrera, Suris. Variational symmetries and pluri-Lagrangian systems in classical mechanics. Journal of Nonlinear Mathematical Physics, 24(sup1), 121-145, 2017.

[3] Suris. Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geometric Mechanics, 2013.

[4] Lobb, Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009.

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Integrable systems

An integrable system is (a system of) nonlinear differential or difference equation(s), that behaves as if it were linear:

- Solvability (in some sense)
- Superposition principle (for special solutions)
- Rich hidden structure explaining nice behavior

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Our perspective

An equation is integrable if it is part of a "sufficiently large" system of "compatible" equations.

In Mechanics: A Hamiltonian system with Hamilton function $H : T^*Q \simeq \mathbb{R}^{2N} \to \mathbb{R}$ with respect to a Poisson bracket $\{\cdot,\cdot\}$ is Liouville-Arnold integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \ldots H_N$ such that

$$\{H_i, H_j\} = 0.$$

Main question

Equations in an integrable hierarchy must be compatible. In the Hamiltonian picture this means:

$$\{H_i, H_j\} = 0.$$

What about the Lagrangian side?

Is there a variational description of an integrable hierarchy?

Lagrangian Mechanics

Lagrange function: $L : \mathbb{R}^{2N} \cong TQ \to \mathbb{R} : (q, \dot{q}) \mapsto L(q, \dot{q})$.

Solutions are curves $q(t)$ that minimize (or are critical points of) the action

$$\int_{t_0}^{t_1} L(q(t), \dot{q}(t)) \, \mathrm{d}t$$

where the integration interval $[t_0, t_1]$ and the boundary values $q(t_0)$ and $q(t_1)$ are fixed.

$$\begin{aligned} 0 = \delta \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) \, \mathrm{d}t &= \int_{t_0}^{t_1} \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \, \mathrm{d}t \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} \right) \delta q \, \mathrm{d}t + \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \bigg|_{t_0}^{t_1} \end{aligned}$$

Euler-Lagrange Equation: $\displaystyle \frac{\delta L}{\delta q} := \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} = 0.$

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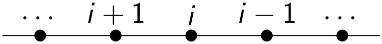
Example: Toda Lattice

Configuration variable $q \in \mathbb{R}^n$, positions of n particles on a line.



Displacements from equilibrium: $\dots, q_{i+1}, q_i, q_{i-1}, \dots$.

Lagrangian: $L(q, \dot{q}) = \sum_{i=1}^{n} \left(\frac{1}{2} \dot{q}_i^2 - e^{q_{i+1} - q_i} \right),$

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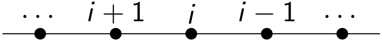

Displacements from equilibrium: $\ldots, q_{i+1}, q_i, q_{i-1}, \ldots$.

Lagrangian: $L(q, \dot{q}) = \displaystyle\sum_{i=1}^{n} \left(\frac{1}{2} \dot{q}_i^2 - e^{q_{i+1} - q_i} \right)$,

The Euler-Lagrange equations are

$$0 = \frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} = e^{q_i - q_{i-1}} - e^{q_{i+1} - q_i} - \ddot{q}_i,$$

so the dynamics are determined by

$$\ddot{q}_i = e^{q_i - q_{i-1}} - e^{q_{i+1} - q_i}$$

This is the first of a formally infinite hierarchy of ODEs;
n of them are independent.

Lagrangian PDEs

Lagrangian $L(v, v_t, v_x, v_{tt}, v_{xt}, v_{xx}, \ldots)$, action $\mathcal{S} = \int L \,\mathrm{d}x\,\mathrm{d}t$.

Look for a function v that is a critical point of the action, i.e. for arbitrary infinitesimal variations δv:

$$\begin{aligned}
0 = \delta\mathcal{S} = \int \delta L \,\mathrm{d}x\,\mathrm{d}t &= \int \sum_I \frac{\partial L}{\partial v_I} \delta v_I \,\mathrm{d}x\,\mathrm{d}t \\
&= \int \sum_I (-1)^{|I|} \left(\mathrm{D}_I \frac{\partial L}{\partial v_I} \right) \delta v \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

Multi-index notation: $I = (i_1, \ldots, i_k)$, $D_I = \dfrac{\mathrm{d}^{i_1}}{\mathrm{d}^{i_1} t_1} \cdots \dfrac{\mathrm{d}^{i_k}}{\mathrm{d}^{i_k} t_k}$ and $v_I = D_I v$.

Euler-Lagrange equation

$$\frac{\delta L}{\delta v} := \sum_I (-1)^{|I|} \mathrm{D}_I \frac{\partial L}{\partial v_I} = 0.$$

Example: KdV equation

Lagrangian: $L = \frac{1}{2} v_x v_t - v_x^3 - \frac{1}{2} v_x v_{xxx}$

Euler-Lagrange Equation:

$$\begin{aligned} 0 = \frac{\delta L}{\delta v} &= \sum_I (-1)^{|I|} \mathrm{D}_I \frac{\partial L}{\partial v_I} \\ &= \frac{\partial L}{\partial v} - \mathrm{D}_t \frac{\partial L}{\partial v_t} - \mathrm{D}_x \frac{\partial L}{\partial v_x} + \mathrm{D}_{xx} \frac{\partial L}{\partial v_{xx}} - \mathrm{D}_{xxx} \frac{\partial L}{\partial v_{xxx}} + \dots \\ &= -v_{xt} + 6 v_x v_{xx} + v_{xxxx} \end{aligned}$$

Substitute $u = v_x$ to find the Korteweg-de Vries equation

$$u_t = 6 u u_x + u_{xxx}.$$

Or integrate to find the Potential Korteweg-de Vries equation

$$v_t = 3 v_x^2 + v_{xxx}.$$

First of an infinite hierarchy of commuting PDEs

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1D Pluri-Lagrangian systems

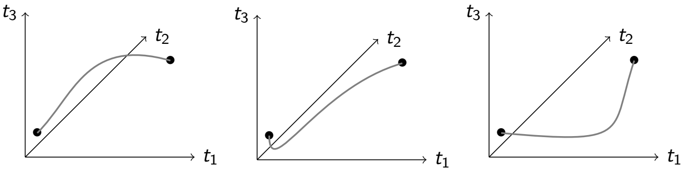
If there are N commuting flows, instead of considering curves $q_i : \mathbb{R} \to Q : t_i \mapsto q(t_i)$, consider a field on multi-time $\mathbb{R}^N$,

$$u : \mathbb{R}^N \to Q : (t_1, \ldots, t_N) \mapsto u(t_1, \ldots, t_N)$$

Definition. A field $u : \mathbb{R}^N \mapsto Q$ is a solution of the pluri-Lagrangian problem for the 1-form,

$$\mathcal{L} = \sum_i L_i[u] \, \mathrm{d}t_i$$

if the action $\int_\Gamma \mathcal{L}$ is critical on all smooth curves Γ in $\mathbb{R}^N$, with respect to variation δu that vanish on the endpoints.



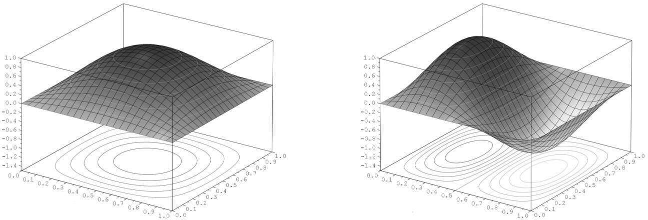
2D Pluri-Lagrangian systems

Definition

A field $u : \mathbb{R}^N \mapsto Q$ is a solution of the pluri-Lagrangian problem for the Lagrangian 2-form,

$$\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j .$$

if the action $\int_\Gamma \mathcal{L}$ is critical on all smooth 2-surfaces Γ in $\mathbb{R}^N$.



Typically, $t_1 = x$ and other t_i correspond to commuting flows.

Usually, $Q = \mathbb{R}$ or $Q = \mathbb{C}$.

Closedness of the Lagrangian form

One could require additionaly that $\mathcal{L}$ is closed on solutions.

Then the action is not just critical on every curve/surface, but also takes the same value on every curve/surface.

We do not take this as part of the definition, because one can show

Proposition

$\mathrm{d}\mathcal{L}$ is constant on the set of solutions.

Often there is a trivial solution that forces this constant to be 0.

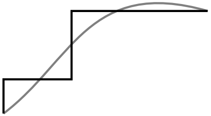
The (almost-) closedness property is a key property to relate the pluri-Lagrangain theory to other notions of integrability.

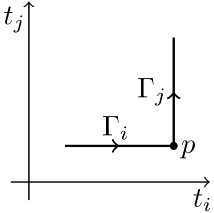
Multi-time Euler-Lagrange equations: 1D case

Consider a Lagrangian 1-form $\mathcal{L} = \sum_i L_i[u] \, \mathrm{d}t_i$

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.

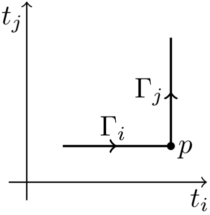




Variations are local, so it is sufficient to look at a general L-shaped curve $\Gamma = \Gamma_i \cup \Gamma_j$.

Multi-time Euler-Lagrange equations: 1D case

The variation of the action on S_i is



$$\begin{aligned}\delta \int_{\Gamma_i} L_i \,\mathrm{d}t_i &= \int_{\Gamma_i} \sum_I \frac{\partial L_i}{\partial u_I} \delta u_I \,\mathrm{d}t_i \\ &= \int_{\Gamma_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta u_I} \delta u_I \,\mathrm{d}t_i + \sum_I \frac{\delta_i L_i}{\delta u_{It_i}} \delta u_I \bigg|_p ,\end{aligned}$$

where I denotes a multi-index, and

$$\frac{\delta_i L_i}{\delta u_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial L_i}{\partial u_{It_i^\alpha}} = \frac{\partial L_i}{\partial u_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial u_{It_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_i}{\partial u_{It_i^2}} - \dots$$

Multi-time Euler-Lagrange equations for curves, $\mathcal{L} = \sum_i L_i[u] \,\mathrm{d}t_i$

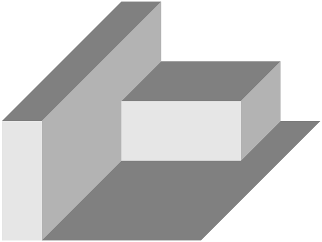
$$\frac{\delta_i L_i}{\delta u_I} = 0 \qquad \forall I \not\ni t_i \qquad \text{and} \qquad \frac{\delta_i L_i}{\delta u_{It_i}} = \frac{\delta_j L_j}{\delta u_{It_j}} \qquad \forall I,$$

Multi-time Euler-Lagrange equations: 2D case

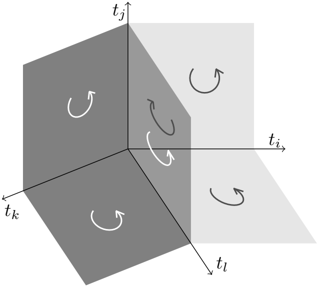
Consider a Lagrangian two-form

$$\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j .$$

It is sufficient to look at stepped surfaces and their elementary corners.



An arbitrary number k of planes can meet in one point, forming a k-flower.
A k-flower can be decomposed into 3-flowers.



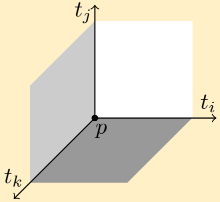
Multi-time Euler-Lagrange equations: 2D case

Multi-time EL equations for surfaces, $\mathcal{L} = \sum_{i,j} L_{ij}[u]\,\mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I.$$

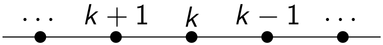


Where

$$\begin{aligned}
\frac{\delta_{ij} L_{ij}}{\delta u_I} &= \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial u_{It_i^\alpha t_j^\beta}} \\
&= \frac{\partial L_{ij}}{\partial u_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_{ij}}{\partial u_{It_i}} - \frac{\mathrm{d}}{\mathrm{d}t_j} \frac{\partial L_{ij}}{\partial u_{It_j}} \\
&\qquad + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_{ij}}{\partial u_{It_i t_i}} + \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\mathrm{d}}{\mathrm{d}t_j} \frac{\partial L_{ij}}{\partial u_{It_i t_j}} + \frac{\mathrm{d}^2}{\mathrm{d}t_j^2} \frac{\partial L_{ij}}{\partial u_{It_j t_j}} - \dots
\end{aligned}$$

Example: Toda hierarchy

$\mathcal{L} = L_1[q]\mathrm{d}t_1 + L_2[q]\mathrm{d}t_2$



$$L_1[q] = \sum_k \tfrac{1}{2}(q_k)_{t_1}^2 - e^{q_k - q_{k-1}}$$

$$L_2[q] = \sum_k (q_k)_{t_1}(q_k)_{t_2} + \tfrac{1}{3}(q_k)_{t_1}^3 + ((q_{k-1})_{t_1} + (q_k)_{t_1})e^{q_k - q_{k-1}}$$

$$\frac{\delta_1 L_1}{\delta q_k} = 0 \quad \Rightarrow \quad (q_k)_{t_1 t_1} = e^{q_{k+1} - q_k} - e^{q_k - q_{k-1}}$$

$$\frac{\delta_2 L_2}{\delta q_k} = 0 \quad \Rightarrow \quad (q_k)_{t_1 t_2} = ((q_k)_{t_1} + (q_{k+1})_{t_1})e^{q_{k+1} - q_k} - ((q_{k-1})_{t_1} + (q_k)_{t_1})e^{q_k - q_{k-1}}$$

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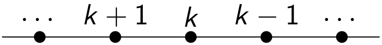

$$L_1[q] = \sum_k \tfrac{1}{2}(q_k)_{t_1}^2 - e^{q_k - q_{k-1}}$$

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$$\frac{\delta_2 L_2}{\delta (q_k)_{t_1}} = 0 \quad \Rightarrow \quad (q_k)_{t_2} = -(q_k)_{t_1}^2 + e^{q_{k+1} - q_k} - e^{q_k - q_{k-1}}$$

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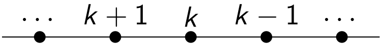

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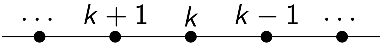

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Pluri-Lagrangian formalism produces evolutionary equations!

Example: Potential KdV hierarchy

$$v_{t_2} = g_2[v] = v_{xxx} + 3v_x^2,$$

$$v_{t_3} = g_3[v] = v_{xxxxx} + 10 v_x v_{xxx} + 5 v_{xx}^2 + 10 v_x^3,$$

where we identify $t_1 = x$.

The differentiated equations $v_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x} g_i[v]$ are Lagrangian with

$$L_{12} = \frac{1}{2} v_x v_{t_2} - \frac{1}{2} v_x v_{xxx} - v_x^3,$$

$$L_{13} = \frac{1}{2} v_x v_{t_3} - v_x v_{xxxxx} - 2 v_{xx} v_{xxxx} - \frac{3}{2} v_{xxx}^2 + 5 v_x^2 v_{xxx} + 5 v_x v_{xx}^2 + \frac{5}{2} v_x^4.$$

Example: Potential KdV hierarchy

$$v_{t_2} = g_2[v] = v_{xxx} + 3v_x^2,$$

$$v_{t_3} = g_3[v] = v_{xxxxx} + 10 v_x v_{xxx} + 5 v_{xx}^2 + 10 v_x^3,$$

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We choose the coefficient L_{23} of

$$\mathcal{L} = L_{12}[u] \, \mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}[u] \, \mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}[u] \, \mathrm{d}t_2 \wedge \mathrm{d}t_3$$

such that the pluri-Lagrangian 2-form is closed on solutions (nontrivial task!). It is of the form

$$L_{23} = \frac{1}{2} \left(v_{t_2} g_3[v] - v_{t_3} g_2[v] \right) + p_{23}[v].$$

Example: Potential KdV hierarchy

- The equations $\dfrac{\delta_{12} L_{12}}{\delta v} = 0$ and $\dfrac{\delta_{13} L_{13}}{\delta v} = 0$ yield

$$v_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x} g_2[v] \qquad \text{and} \qquad v_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x} g_3[v].$$

- The equations $\dfrac{\delta_{12} L_{12}}{\delta v_x} = \dfrac{\delta_{32} L_{32}}{\delta v_{t_3}}$ and $\dfrac{\delta_{13} L_{13}}{\delta v_x} = \dfrac{\delta_{23} L_{23}}{\delta v_{t_2}}$ yield

$$v_{t_2} = g_2 \qquad \text{and} \qquad v_{t_3} = g_3,$$

  the evolutionary equations!
- All other multi-time EL equations are corollaries of these.

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[2] Petrera, Suris. Variational symmetries and pluri-Lagrangian systems in classical mechanics. Journal of Nonlinear Mathematical Physics, 24(sup1), 121-145, 2017.

Noether's theorem

Consider a mechanical Lagrangian $L(q, q_t)$.

We say that a (generalized) vector field $V(q, q_t)\frac{\partial}{\partial q}$ is a variational symmetry if there exists a function $F(q, q_t)$, called the flux, such that

$$\mathrm{D}_V L(q, q_t) - \mathrm{D}_t F(q, q_t) = 0,$$

where

$$\mathrm{D}_t = q_t \frac{\partial}{\partial q} + q_{tt} \frac{\partial}{\partial q_t} \qquad \text{and} \qquad D_V = V(q, q_t)\frac{\partial}{\partial q} + (\mathrm{D}_t V(q, q_t))\frac{\partial}{\partial q_t}.$$

Noether's Theorem

If $V(q, q_t)$ is a variational symmetry with flux $F(q, q_t)$, then

$$J(q, q_t) = \frac{\partial L(q, q_t)}{\partial q_t} \cdot V(q, q_t) - F(q, q_t)$$

is an integral of motion.

From variational symmetries to pluri-Lagrangian structure

If we have a variational symmetry V with Noether integral J, then with

$$L_1(q, q_{t_1}) = L(q, q_{t_1})$$
$$L_2(q, q_{t_1}, q_{t_2}) = \frac{\partial L(q, q_{t_1})}{\partial q_{t_1}} \cdot (q_{t_2} - V(q, q_{t_1})) + F(q, q_{t_1})$$

the pluri-Lagrangian one-form

$$\mathcal{L} = L_1(q, q_{t_1})\,\mathrm{d}t_1 + L_2(q, q_{t_1}, q_{t_2})\,\mathrm{d}t_2$$

produces the equations of motion

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L}{\partial q_{t_1}} = 0 \qquad \text{and} \qquad q_{t_2} = V(q, q_{t_1})$$

By construction we have $\mathrm{D}_{t_2} L_1 - \mathrm{D}_{t_1} L_2 = 0$ on solutions
$\Rightarrow \mathcal{L}$ is closed on solutions.

Similar result for PDEs: work in progress.

k commuting variational symmetries $\rightarrow$ pluri-Lagrangian system in $\mathbb{R}^{k+1}$

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[3] Suris. Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geometric Mechanics, 2013.

Simultaneous Legendre transform of the Toda hierarchy

$$L_1[q] = \sum_k \frac{1}{2}(q_k)_{t_1}^2 - e^{q_k - q_{k-1}}$$

$$L_2[q] = \sum_k (q_k)_{t_1}(q_k)_{t_2} + \frac{1}{3}(q_k)_{t_1}^3 + ((q_{k-1})_{t_1} + (q_k)_{t_1})e^{q_k - q_{k-1}}$$

In itself L_2 is degenerate, but the whole system can still be Legendre-transformed: $p_k = (q_k)_{t_1}$ and

$$H_1(q,p) = \sum_k \frac{1}{2}p_k^2 + e^{q_k - q_{k-1}}$$

$$H_2(q,p) = \sum_k -\frac{1}{3}p_k^3 - (p_{k-1} + p_k)e^{q_k - q_{k-1}}$$

with the canonical Poisson bracket

$$\{f,g\} = \sum_{i=1}^N \left(\frac{\partial f}{\partial q_{t_i}} \frac{\partial g}{\partial p_{t_i}} - \frac{\partial f}{\partial p_{t_i}} \frac{\partial g}{\partial q_{t_i}} \right)$$

Closedness of the Lagrangian form

Theorem

$$\mathrm{d}\mathcal{L} = \sum_{i,j} \{H_j, H_i\} \, \mathrm{d}t_i \wedge \mathrm{d}t_j$$

Proof. Hamiltonian side:

$$\begin{aligned} \mathrm{D}_i L_j - \mathrm{D}_j L_i &= \{H_i, L_j\} - \{H_j, L_i\} \\ &= \{H_i, pq_{t_j} - H_j\} - \{H_j, pq_{t_i} - H_i\} \\ &= 2\{H_j, H_i\} - (p_{t_j} q_{t_i} - p_{t_i} q_{t_j}). \end{aligned} \tag{1}$$

Pluri-Lagrangian side $\mathrm{D}_j L_i = p_{t_i} q_{t_j} + p q_{t_i t_j}$, hence

$$\mathrm{D}_i L_j - \mathrm{D}_j L_i = p_{t_j} q_{t_i} - p_{t_i} q_{t_j}. \tag{2}$$

(1) and (2) yield $\mathrm{D}_i L_j - \mathrm{D}_j L_i = \{H_j, H_i\}$. □

Closedness of the Lagrangian form

Theorem

$$\mathrm{d}\mathcal{L} = \sum_{i,j} \{H_i, H_j\} \,\mathrm{d}t_i \wedge \mathrm{d}t_j$$

From the pluri-Lagrangian theory, one can show that $\mathrm{d}\mathcal{L} = \textit{const}$ on solutions
$\Rightarrow$ Poisson brackets of two Hamiltonians are constant
$\Rightarrow$ flows commute.

$\mathcal{L}$ closed on solutions $\Leftrightarrow$ Hamiltonians in involution.

Similar result for PDEs: work in progress.

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[4] Lobb, Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009.

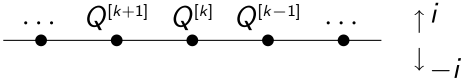
[5] Hietarinta, Joshi, Nijhoff. Discrete systems and integrability. Cambridge university press, 2016.

Example: fully discrete Toda Lattice

An integrable discretization of the Toda lattice is given by

$$\frac{1}{\lambda_i}\left(\exp\left(Q^{[k]} - Q^{[k]}_{-i}\right) - \exp\left(Q^{[k]}_i - Q^{[k]}\right)\right) \\ + \lambda_i\left(\exp\left(Q^{[k+1]}_{-i} - Q^{[k]}\right) - \exp\left(Q^{[k]} - Q^{[k-1]}_i\right)\right) = 0,$$

where the subscripts i and $-i$ denote forward and backward shifts respectively and λ_i is a lattice parameter.



A Lagrangian for the fully discrete Toda lattice is

$$L\left(Q, Q_i, \lambda_i\right) = \frac{1}{\lambda_i}\sum_k \exp\left(Q^{[k]}_i - Q^{[k]}\right) - \lambda_i \sum_k \exp\left(Q^{[k]} - Q^{[k-1]}_i\right)$$

Consider it as a discrete 1-form, depending on the orientation of the edge:

$$L\left(Q_i, Q, \lambda_i\right) = -\frac{1}{\lambda_i}\sum_k \exp\left(Q^{[k]}_i - Q^{[k]}\right) + \lambda_i \sum_k \exp\left(Q^{[k]} - Q^{[k-1]}_i\right)$$

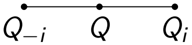
Discrete pluri-Lagrangian 1-forms

$$L\big(Q, Q_i, \lambda_i\big) = \frac{1}{\lambda_i} \sum_k \exp\big(Q_i^{[k]} - Q^{[k]}\big) - \lambda_i \sum_k \exp\big(Q^{[k]} - Q_i^{[k-1]}\big)$$
$$L\big(Q_i, Q, \lambda_i\big) = -L\big(Q, Q_i, \lambda_i\big)$$

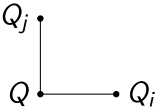
Discrete pluri-Lagrangian principle

Action is critical along any discrete curve in the lattice.

Discrete multi-time Euler-Lagrange equations

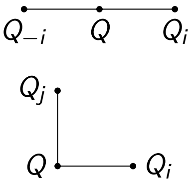


$$\frac{\partial}{\partial Q}\left(L\big(Q_{-i}, Q, \lambda_i\big) + L\big(Q, Q_i, \lambda_i\big)\right) = 0$$



$$\frac{\partial}{\partial Q}\left(L\big(Q_j, Q, \lambda_j\big) + L\big(Q, Q_i, \lambda_i\big)\right) = 0$$

Discrete pluri-Lagrangian 1-forms



$$\frac{\partial}{\partial Q}\Big(L\big(Q_{-i},Q,\lambda_i\big)+L\big(Q,Q_i,\lambda_i\big)\Big)=0$$

$$\frac{\partial}{\partial Q}\Big(L\big(Q_{j},Q,\lambda_j\big)+L\big(Q,Q_i,\lambda_i\big)\Big)=0$$

If Q is a solution we can find P such that

$$\begin{aligned} P^{[k]} &= -\frac{\partial}{\partial Q^{[k]}}L\big(Q_i,Q,\lambda_i\big)=\frac{\partial}{\partial Q^{[k]}}L\big(Q_{-i},Q,\lambda_i\big) \\ &= \frac{1}{\lambda_i}\exp\big(Q_i^{[k]}-Q^{[k]}\big)+\lambda_i\exp\big(Q^{[k+1]}-Q_i^{[k]}\big) \end{aligned} \qquad \text{for } i=1,\dots,N$$

Then

$$(Q,P)\mapsto(Q_i,P_i) \qquad \text{for } i=1,\dots,N$$

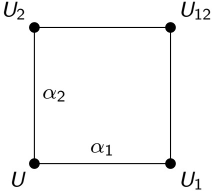
are commuting symplectic maps.

Quad equations

Quad equation:

$$\mathcal{Q}(U, U_1, U_2, U_{12}, \alpha_1, \alpha_2) = 0$$

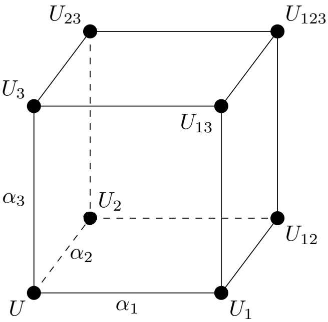
Subscripts of x denote lattice shifts, α_1, α_2 are parameters.
Invariant under symmetries of the square, affine in each of U, U_1, U_2, U_{12}.



Integrability for systems quad equations:
Multi-dimensional consistency of

$$\mathcal{Q}(U, U_i, U_j, U_{ij}, \alpha_i, \alpha_j) = 0,$$

i.e. the three ways of calculating U_{123} give the same result.



Pluri-Lagrangian structure for quad equations

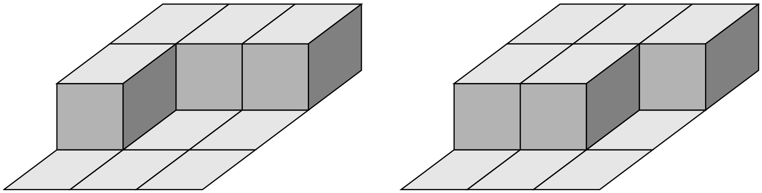
For some discrete 2-form

$$\mathcal{L}(\square_{ij}) = \mathcal{L}(U, U_i, U_j, U_{ij}, \alpha_i, \alpha_j),$$

the action

$$\sum_{\square \in \Gamma} \mathcal{L}(\square)$$

is critical on all 2-surfaces Γ in $\mathbb{N}^N$ simultaneously.



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[6] V. Continuum limits of pluri-Lagrangian systems. arXiv:1706.06830

Continuum limit of the discrete Toda Lattice

Recall the discretization of the Toda lattice

$$\frac{1}{\lambda_1}\left(\exp\left(Q_1^{[k]} - Q^{[k]}\right) - \exp\left(Q^{[k]} - Q_{-1}^{[k]}\right)\right) + \lambda_1\left(\exp\left(Q^{[k]} - Q_1^{[k-1]}\right) - \exp\left(Q_{-1}^{[k+1]} - Q^{[k]}\right)\right) = 0 \qquad (*)$$

Miwa shifts

Skew embedding of the mesh $\mathbb{Z}^N$ into multi-time $\mathbb{R}^N$

Discrete Q is a sampling of the continuous q:

$$Q = Q(\mathbf{n}) = q(t_1, t_2, \dots, t_N),$$

$$Q_i = Q(\mathbf{n} + \mathfrak{e}_i) = q\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \dots, t_N + 2(-1)^N \frac{\lambda_i^N}{N}\right)$$

[Miwa. On Hirota's difference equations. Proceedings of the Japan Academy A. 1982]

Write $(*)$ in terms of q and expand in λ_1.

Continuum limit of the discrete Toda Lattice

Series expansion of the lattice equation

$$
\begin{aligned}
0 = &\left(-\exp\left(q^{[k+1]} - q^{[k]}\right) + \exp\left(q^{[k]} - q^{[k-1]}\right) + q_{11}^{[k]}\right)\lambda_1 \\
&+ \left(\exp\left(q^{[k+1]} - q^{[k]}\right) q_1^{[k+1]} - \exp\left(q^{[k]} - q^{[k-1]}\right) q_1^{[k-1]} + q_1^{[k]} q_{11}^{[k]} - q_{12}^{[k]}\right)\lambda_1^2 \\
&+ \ldots,
\end{aligned}
$$

Solve order by order:

$$
\begin{aligned}
q_{11}^{[k]} &= \exp\left(q^{[k+1]} - q^{[k]}\right) - \exp\left(q^{[k]} - q^{[k-1]}\right) \\
q_{12}^{[k]} &= \frac{\mathrm{d}}{\mathrm{d}t_1}\left(\left(q_1^{[k]}\right)^2 + \exp\left(q^{[k+1]} - q^{[k]}\right) + \exp\left(q^{[k]} - q^{[k-1]}\right)\right) \\
&\vdots
\end{aligned}
$$

Whole hierarchy from single quad equation!

Continuum limit of a Lagrangian 1-form

$$
\begin{array}{ccc}
L(Q, Q_1, \lambda_1) & & \\
\Big\downarrow & \text{Miwa shifts, Taylor expansion} & \\
\mathcal{L}_{\mathrm{disc}}([q], \lambda_1,) & & \\
\Big\downarrow & \text{Euler-Maclaurin formula: action sum} \to \text{action integral} & \\
\mathcal{L}_{\mathrm{Miwa}}([q], \lambda_1) & = & \displaystyle\sum_{i=1}^{\infty} (-1)^i \frac{\lambda_1^i}{i} \mathcal{L}_i[q] \\
\Big\downarrow & & \Big\downarrow \\
\displaystyle\sum_{i=1}^{\infty} \mathcal{L}_{\mathrm{Miwa}}([q], \lambda_i)\, \eta_i & \approx & \displaystyle\sum_{i=1}^{\infty} \mathcal{L}_i[q]\, \mathrm{d}t_i
\end{array}
$$

where $\approx$ denotes equality with a defect of order λ_i^{n+1}.

Continuum limit of a Lagrangian 2-form

$L(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2)$ Suitable choice $\Rightarrow$ leading order cancellation

$$\Big\downarrow \text{ Miwa shifts, Taylor expansion}$$

$$\mathcal{L}_{\mathrm{disc}}([u], \lambda_1, \lambda_2)$$

$$\Big\downarrow \text{ Euler-Maclaurin formula}$$

$$\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2) \quad = \quad \sum_{i,j=1}^{\infty} (-1)^{i+j} 4 \frac{\lambda_1^i}{i} \frac{\lambda_2^j}{j} \mathcal{L}_{i,j}[u]$$

$$\Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow$$

$$\sum_{1 \leq i < j \leq N} \mathcal{L}_{\mathrm{Miwa}}([u], \lambda_i, \lambda_j)\, \eta_i \wedge \eta_j \qquad \approx \qquad \sum_{1 \leq i < j \leq N} \mathcal{L}_{i,j}[u]\, \mathrm{d}t_i \wedge \mathrm{d}t_j$$

where $\approx$ denotes equality with a defect of order λ_i^{n+1}.

Summary

1. Background
2. Pluri-Lagrangian systems [1]
3. Relation to variational symmetries [2]
4. Relation to Hamiltonian formalism [3]
5. Discrete pluri-Lagrangian systems [4,5,. . .]
6. Continuum limits [6]

[1] Suris, V. On the Lagrangian structure of integrable hierarchies. In Advances in Discrete Differential Geometry, Springer, 2016.

[2] Petrera, Suris. Variational symmetries and pluri-Lagrangian systems in classical mechanics. Journal of Nonlinear Mathematical Physics, 24(sup1), 121-145, 2017.

[3] Suris. Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geometric Mechanics, 2013.

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Summary

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Thank you for your attention!

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