What is... an integrable system?

Mats Vermeeren

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Discretization in Geometry and Dynamics SFB Transregio 109

Berlin Mathematical School

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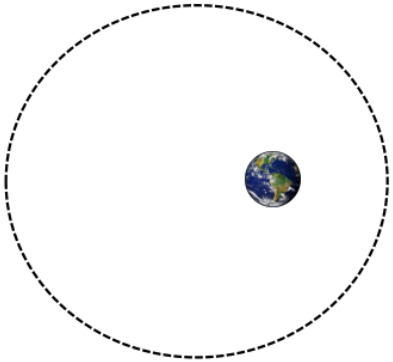
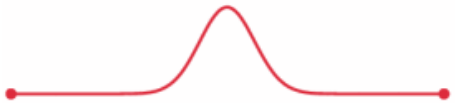
The "system" in "integrable system" can be

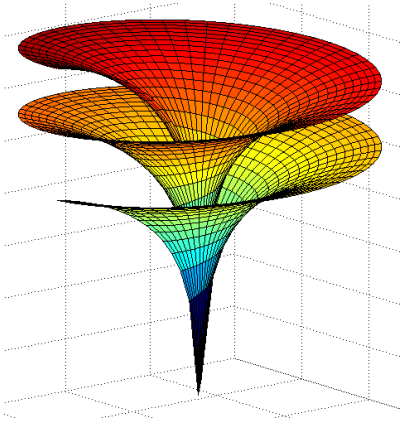
- a set of differential equations (ordinary or partial)



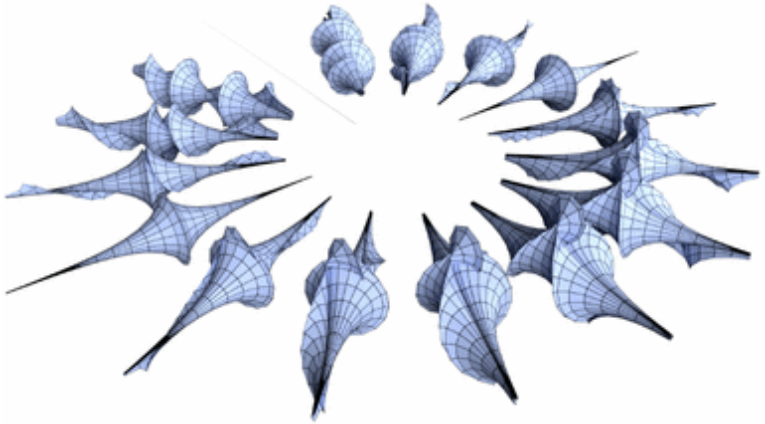


https://sites.google.com/site/ablowitz/

- a set of difference equations
- a geometric object, e.g. a constant negative curvature surface



en.wikipedia.org/wiki/Dini%27s_surface



https://doi.org/10.1007/s00454-016-9802-6

Usually discribed by differential/difference equations.

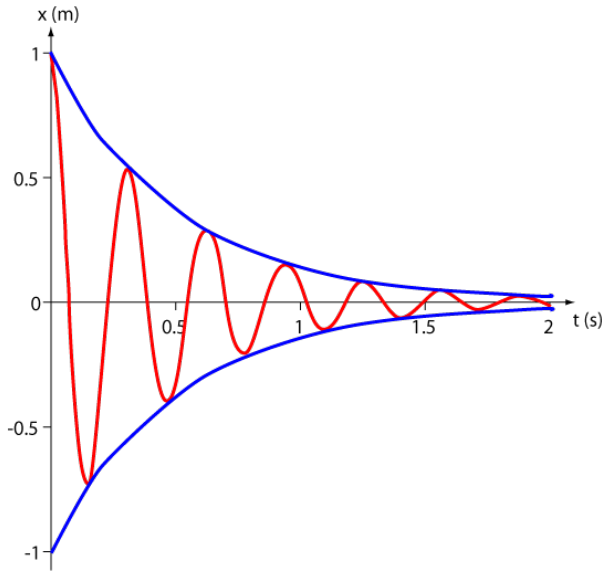
Linear vs nonlinear differential equations

Linear equations are exactly solvable:

$$\ddot{x}(t) = -x(t) - \beta\dot{x}(t)$$

$$\downarrow$$

$$x(t) = e^{-\frac{\beta}{2}t}\cos\left(\sqrt{1-\frac{\beta}{4}}t\right)$$



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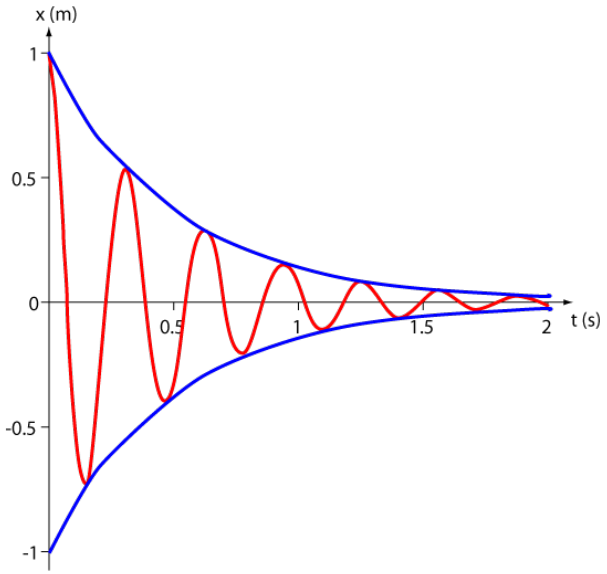
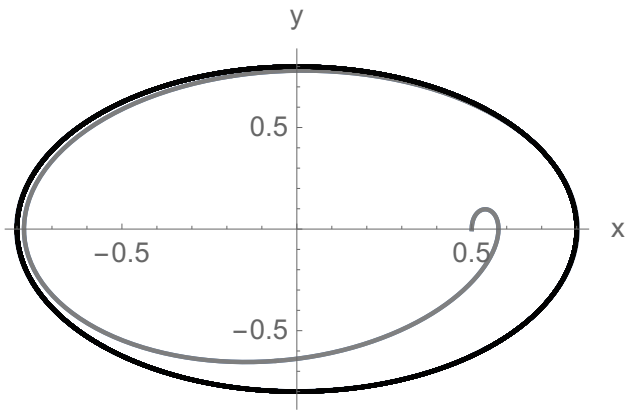
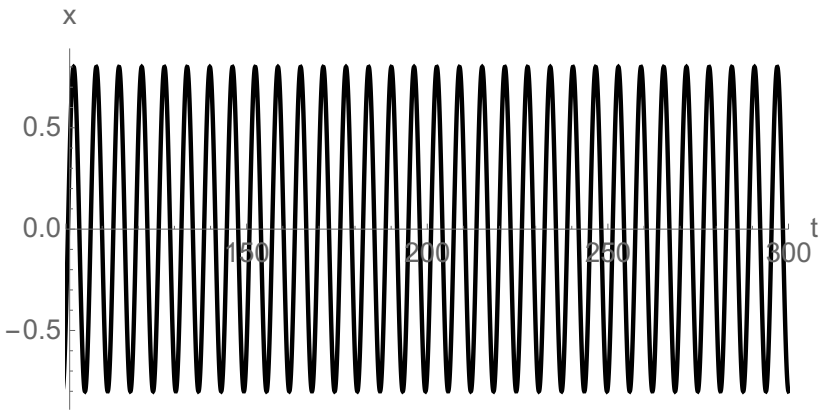


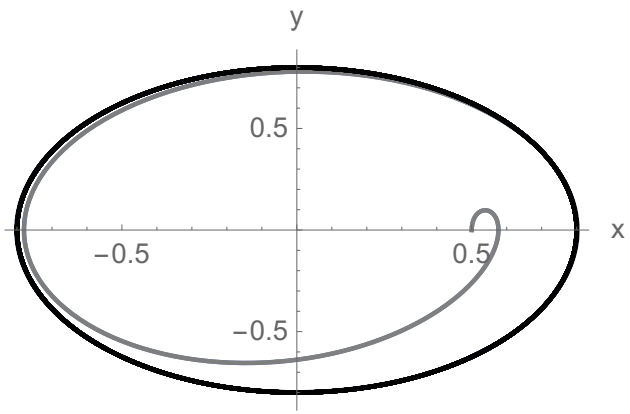
Even if we add a nonlinear forcing, the (limit) behavior is very simple.

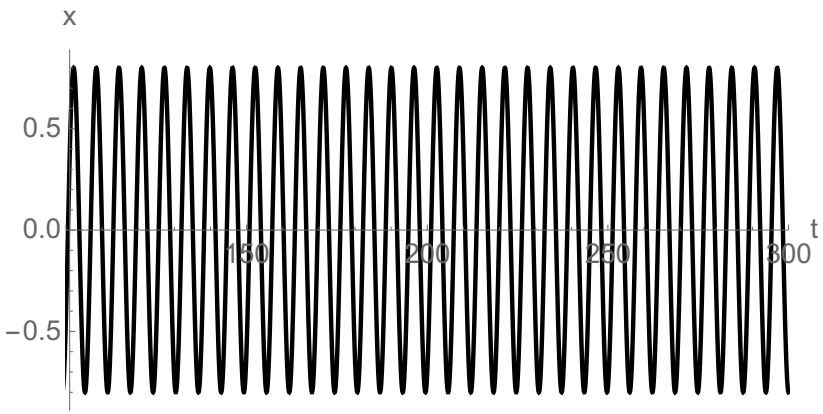




$$\ddot{x}(t) = -\dot{x}(t) - x(t) + 0.8\cos t$$

Linear vs nonlinear differential equations





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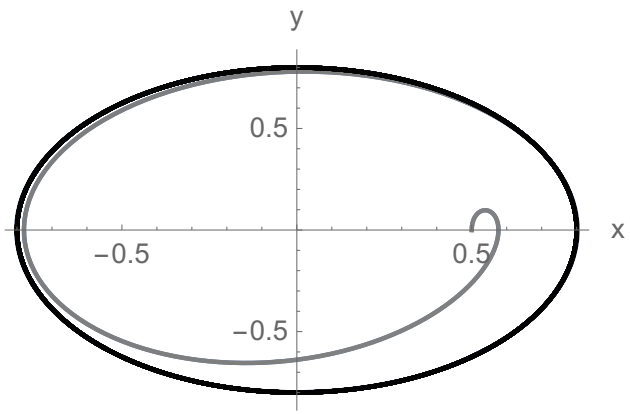

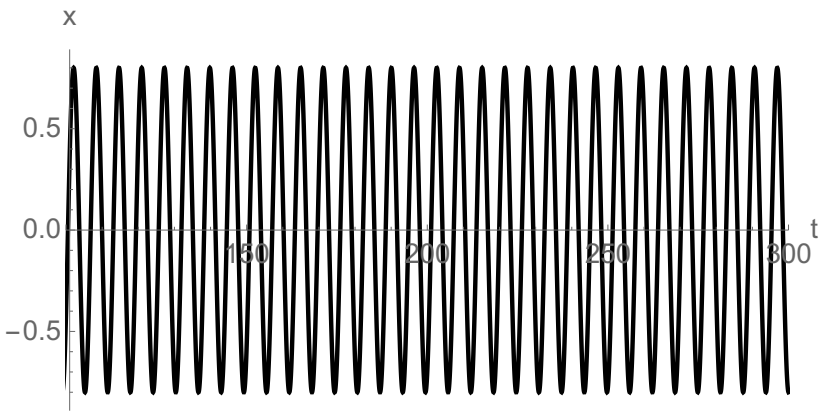

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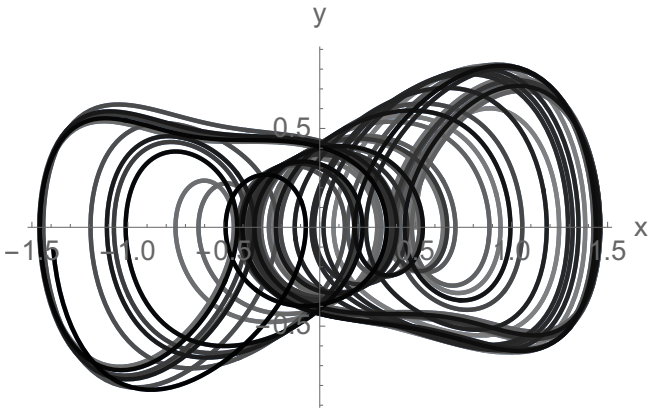

$$\ddot{x}(t) = -\dot{x}(t) + x(t) - x(t)^3 + 0.8\cos t$$

Linear vs nonlinear differential equations

Linear equations are boring

https://commons.wikimedia.org/wiki/File:Wave_packet_(no_dispersion).gif

Nonlinear equations are difficult

http://www.physics.umb.edu/Staff/olchanyi_research/images/saw-Gordon__movie.gif

Integrable systems: nonlinear equations that pretend to be linear

https://commons.wikimedia.org/wiki/Category:Solitons#/media/File:KdV_Solitons2.gif

"Integrable"

Heuristic I

A nonlinear system is integrable if it behaves almost like a linear system.

If an equation is related to a linear system, it can often be solved exactly.

Historic meaning

- 19th century: exactly solvable.
- Modern: surprising structure, that can help solve the system.

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Hamiltonian Systems

Hamilton function

$$H : \mathbb{R}^{2N} \cong T^*Q \to \mathbb{R} : (q, p) \mapsto H(q, p)$$

determines dynamics:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

If $H = \frac{1}{2m}p^2 + U(q)$, then we find Newton's laws:

$$\dot{q} = \frac{1}{m}p \qquad \text{and} \qquad \dot{p} = -\nabla U(q)$$

Geometric interpretation:

- Phase space T^*Q with canonical symplectic 2-form ω
- flow along vector field X_H determined by $\iota_{X_H}\omega = \mathrm{d}H$
- the flows consists of symplectic maps and preserves H.

Poisson Brackets

Poisson bracket of two functionals on T^*Q:

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

Dynamics of a Hamiltonian system:

$$\dot{q}_i = \{q_i, H\}, \qquad \dot{p}_i = \{p_i, H\}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} f(q, p) = \{f(q, p), H\}$$

Properties:

$$\begin{aligned}
\text{anti-symmetry:}\ & \{f, g\} = -\{g, f\} \\
\text{bilinearity:}\ & \{f, g + \lambda h\} = \{f, g\} + \lambda\{f, h\} \\
\text{Leibniz property:}\ & \{f, gh\} = \{f, g\}h + g\{f, h\} \\
\text{Jacobi identity:}\ & \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0
\end{aligned}$$

Liouville-Arnold integrability

What if $H(p,q) = H(p)$?

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = 0 \qquad \text{and} \qquad \dot{q}_j = \frac{\partial H}{\partial p_j} = \omega_j(p) = \mathrm{const}$$

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A Hamiltonian system with Hamilton function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville-Arnold integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \ldots H_N$ such that $\{H_i, H_j\} = 0$.

This implies that

- each H_i is a conserved quantity for all flows.
- the dynamics is confined to a leaf of the foliation $\{H_i = \mathrm{const}\}$.
- the flows commute.
- There exists a symplectic change of variables $(p,q) \mapsto (\bar{p}, \bar{q})$ such that

$$H(p,q) = \bar{H}_i(\bar{p})$$

Liouville-Arnold integrable systems evolve linearly in these variables!

$(\bar{p}, \bar{q})$ are called action-angle variables.

Example: Kepler problem

Physical Hamiltonian: energy

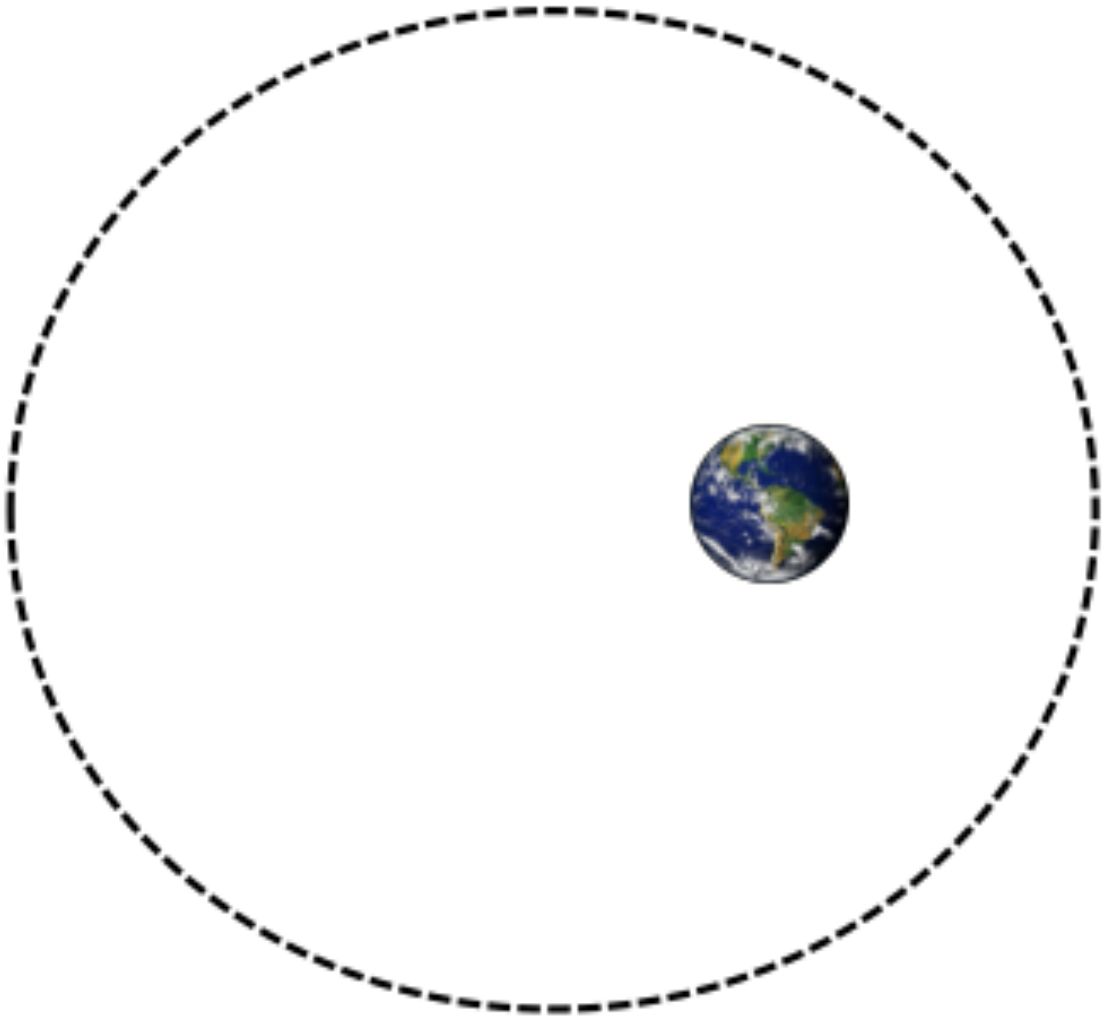
$$H_2 = \frac{1}{2}|p|^2 - \frac{1}{|q|}$$

$$\Rightarrow \begin{cases} \dot{q} = p \\ \dot{p} = -\frac{q}{|q|^3} \end{cases}$$

Additional Hamiltonian: angular momentum

$$H_1 = p^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} q$$

$$\Rightarrow \begin{cases} \dot{q} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} q \\ \dot{p} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p \end{cases}$$



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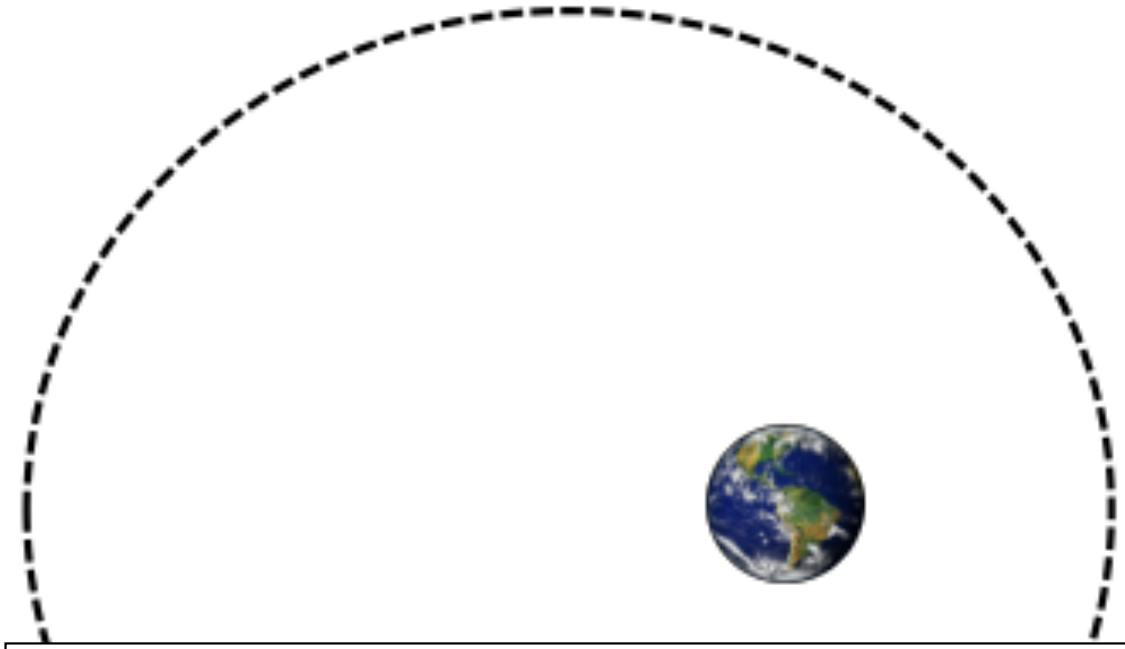

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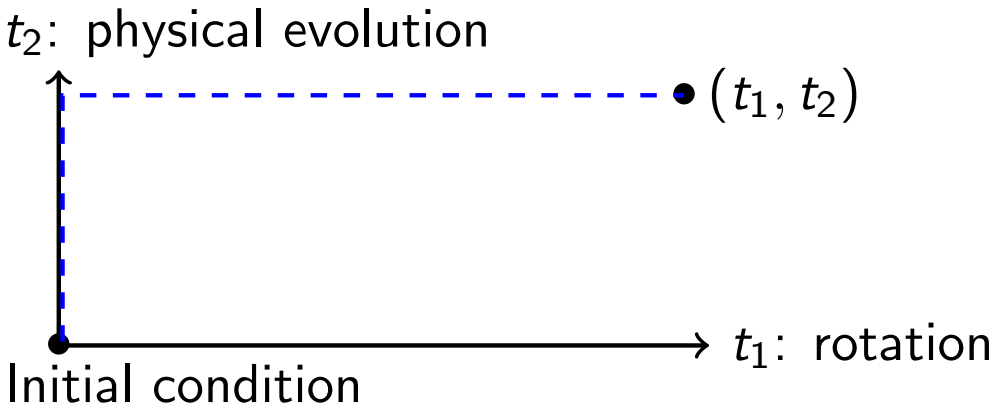
q and p are functions of two times (t_1, t_2).



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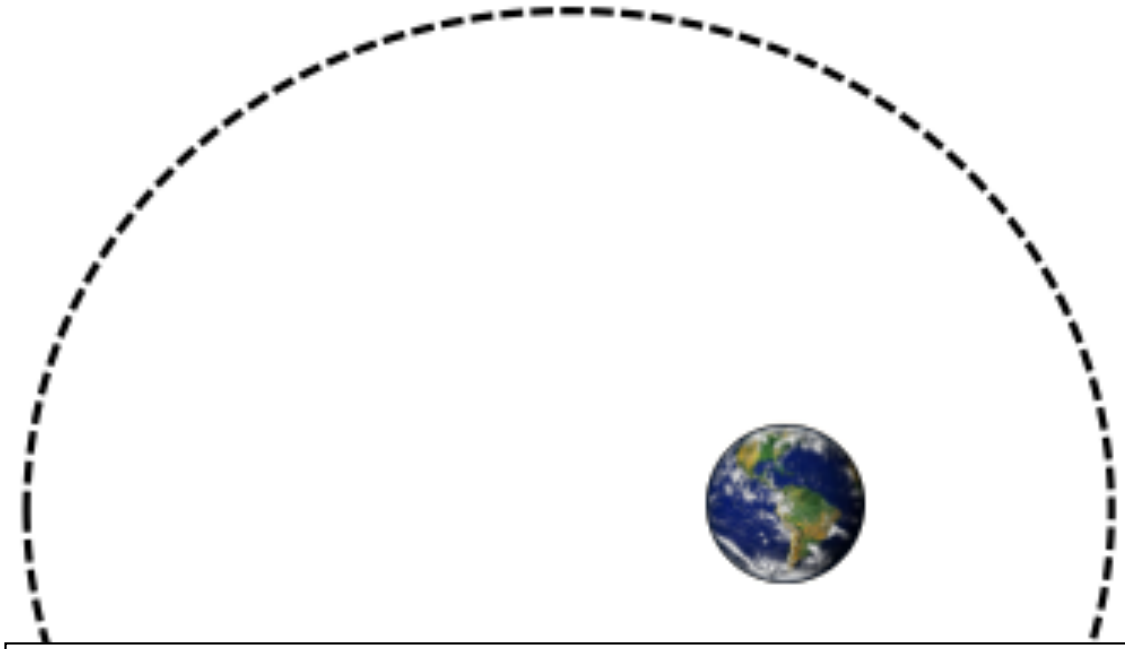

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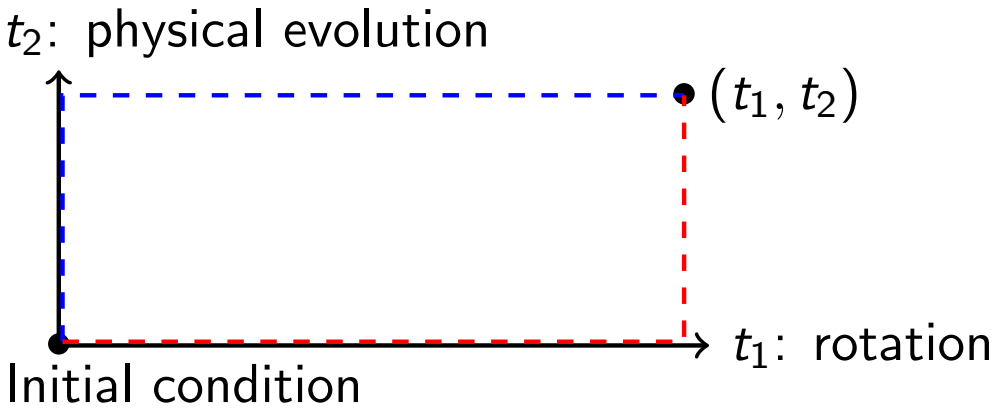

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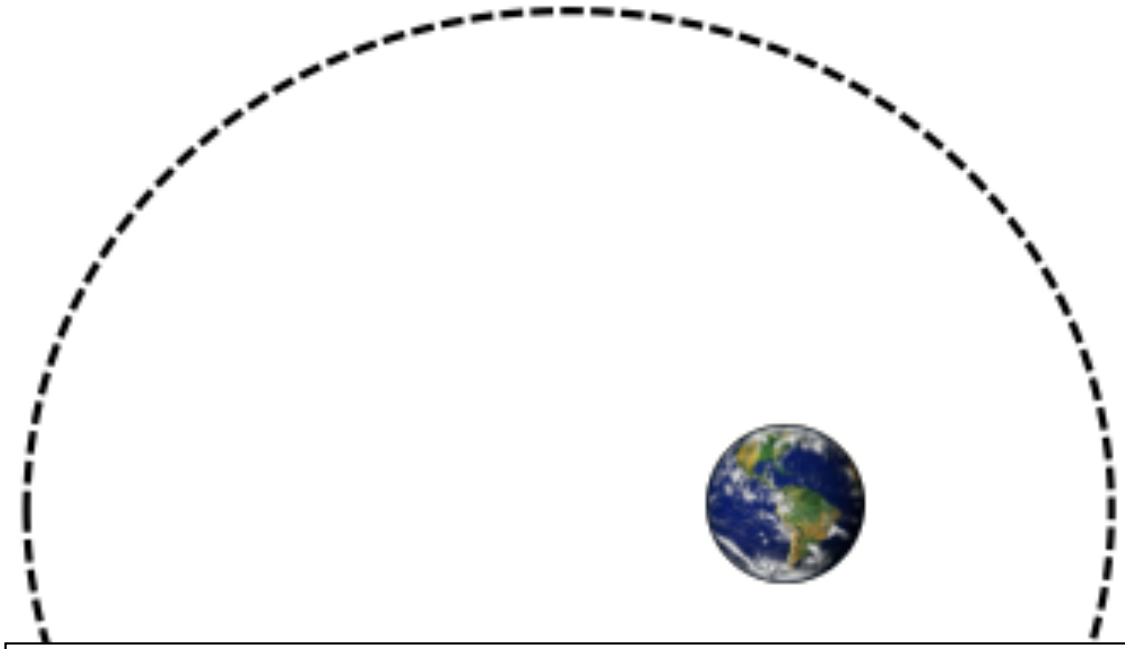

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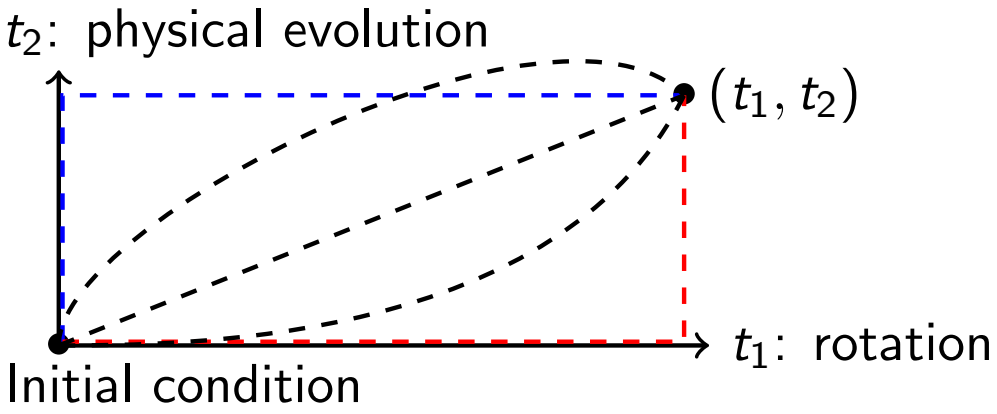

More heuristics

Heuristic II

A system is integrable if it has many conserved quantities
(and these conserved quantities are in involution)

Liouville-Arnold: conserved quantities lead to hidden linear structure

Heuristic III

A system is integrable if it is part of a sufficiently large family of compatible equations

Noether theorem: symmetries lead to conserved quantities
(at least in variational systems)

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Lax Pairs

A Lax Pair consists of two matrices (or operators) L and P,

- depending on the dynamical variables,
- acting on some auxiliary space,
- such that
$$\frac{\mathrm{d}}{\mathrm{d}t}L = [P, L] \qquad := PL - LP \tag{$*$}$$
is equivalent to the equations of motion.

Equation $(*)$ represents the compatibility of

- the eigenvalue problem $L(t)\phi(t) = \lambda\phi(t)$,
- and the linear equation $\dfrac{\mathrm{d}\phi(t)}{\mathrm{d}t} = P(t)\phi(t)$.

Lax Pairs

For all $k \in \mathbb{N}$,

$$\mathrm{tr}(L^k)$$

is a conserved quantity of the system $\frac{\mathrm{d}}{\mathrm{d}t} L = [P, L]$.

proof

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}(L^k) &= \mathrm{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t} L^k\right) \\
&= \mathrm{tr}\left(\frac{\mathrm{d}L}{\mathrm{d}t} L^{k-1} + L\frac{\mathrm{d}L}{\mathrm{d}t} L^{k-2} + \ldots + L^{k-1}\frac{\mathrm{d}L}{\mathrm{d}t}\right) \\
&= \mathrm{tr}\Big((PL - LP)L^{k-1} + L(PL - LP)L^{k-2} + \ldots + L^{k-1}(PL - LP)\Big) \\
&= \mathrm{tr}\Big(PL^k - L^k P\Big) = 0
\end{aligned}$$

■

Compatibility of linear equations leads to conserved quantities.

Example: harmonic oscillator

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & -\frac{1}{2}\omega \\ \frac{1}{2}\omega & 0 \end{pmatrix}$$

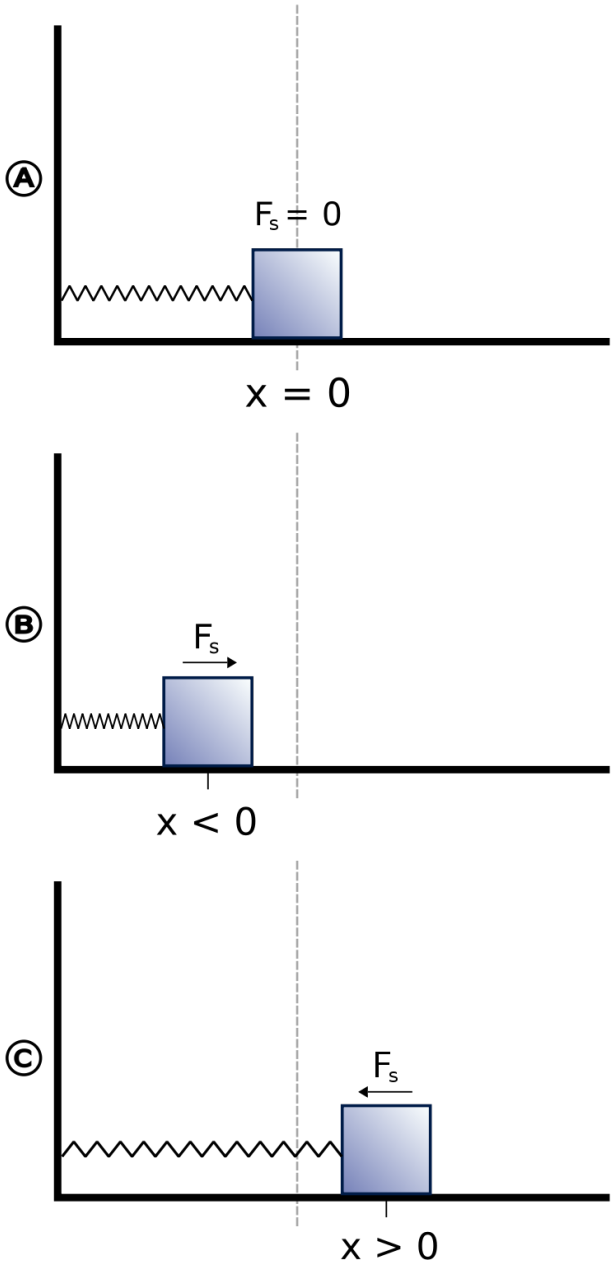
$$\frac{\mathrm{d}}{\mathrm{d}t} L = [P, L] = PL - LP$$

$$\Downarrow$$

$$\begin{pmatrix} \dot{p} & \omega \dot{q} \\ \omega \dot{q} & -\dot{p} \end{pmatrix} = \begin{pmatrix} -\omega^2 q & \omega p \\ \omega p & \omega^2 q \end{pmatrix}$$

$$\Downarrow$$

$$\dot{p} = -\omega^2 q \quad \text{and} \quad \dot{q} = p$$



Example: harmonic oscillator

Conserved quantities:

$$\begin{aligned}
\mathrm{tr}(L) &= 0, \\
\mathrm{tr}(L^2) &= 2p^2 + 2\omega^2 q^2 = 4H, \\
\mathrm{tr}(L^3) &= 0, \\
\mathrm{tr}(L^4) &= 2(p^2 + \omega^2 q^2)^2 = 8H^2, \\
&\dots
\end{aligned}$$

Boring, 1-dimensional linear system.

$4H$ and $8H^2$ are functionally dependent, but for more interesting examples we will get several independent conserved quantities.

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Korteweg-de Vries (KdV) equation

Liouville-Arnold

n degrees of freedom $\Rightarrow$ need n conserved quantities.

PDEs have infinite degrees of freedom.

Indeed, some PDEs, like the KdV equation

$$v_t = v_{xxx} + 6vv_x,$$

have an infinite amount of conserved quantities.

KdV was first derived as a model for water waves in a narrow and shallow canal

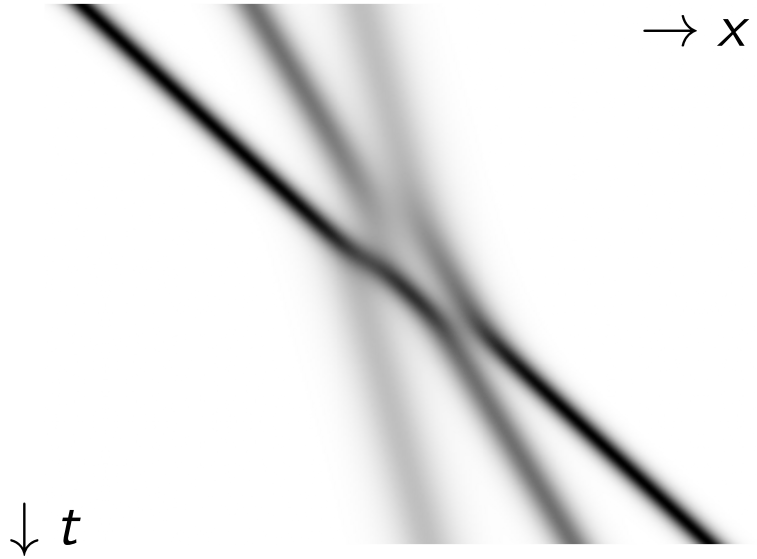
www.ma.hw.ac.uk/solitons/

Water waves described by the KdV equation

https://youtu.be/hfc3IL9gAts

Soliton interaction

https://commons.wikimedia.org/wiki/Category:Solitons#/media/File:KdV_Solitons2.gif



Asymptotic behavior: like superposition, but with phase shift.

A brief history of solitons

1834 John Scott Russell observes Wave of Translation generated by stopping boat in canal
I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure [...] after a chase of one or two miles I lost it in the windings of the channel.

1871 – 1895 Rayleigh, Boussinesq, Korteweg and de Vries develop mathematical models for Russels observation. $\rightarrow$ KdV equation.

1965 Zabusky and Kruskal numerically observe solitons and thier interaction for the KdV equation (and use it to explain the Fermi–Pasta–Ulam–Tsingou experiment).

1967 Gardner, Greene, Kruskal and Miura derive analytic solutions to KdV using the inverse scattering transform (IST). Key idea: scattering data evolve linearly.

1968 Peter Lax understands the IST using pairs of operators.

Lax pairs for the KdV hierarchy

- $\frac{\mathrm{d}}{\mathrm{d}t_k} L = [P_k, L]$ with $L = \partial^2 + v$.
- Only makes sense if $[P_k, L]$ is a function instead of a differential operator.
- This is the case for

$$P_3 = \partial^3 + \frac{3}{2} v \partial + \frac{3}{4} v_x,$$

$$P_5 = \partial^5 + \frac{5}{2} v \partial^3 + \frac{15}{4} v_x \partial^2 + \left(\frac{25}{8} v_{xx} + \frac{15}{8} v^2 \right) \partial + \left(\frac{15}{16} v_{xxx} + \frac{15}{8} v v_x \right)$$

$$\dots$$

Equations:

$$v_{t_3} = v_{xxx} + 6 v v_x$$

$$v_{t_5} = v_{xxxxx} + 20 v_x v_{xx} + 10 v v_{xxx} + 30 v^2 v_x$$

$$\dots$$

Korteweg-de Vries (KdV) hierarchy

One of many integrable hierarchies:

- Nonlinear-Schrödinger
- modified KdV, schwarzian KdV
- Gel'fand Dikii (GD) hierarchies
- Kadomtsev–Petviashvili (KP) hierarchy
- ...

Physical perspective

One physical equation with infinitely many symmetries and hence infinitely many conservation laws.

Mathematical perspective

A hierarchy of equations that are symmetries of each other.

Korteweg-de Vries (KdV) hierarchy

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Physical perspective

One physical equation with infinitely many symmetries and hence infinitely many conservation laws.

Mathematical perspective

A hierarchy of equations that are symmetries of each other.

Recall heuristic III

A system is integrable if it is part of a sufficiently large family of compatible equations.

Hamiltonian structure of the KdV hierarchy

$$v_t = v_{t_3} = v_{xxx} + 6vv_x \qquad \Leftrightarrow \qquad v_t = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta H}{\delta v},$$

where $H = -\frac{1}{2}v_x^2 + v^3$ and

$$\frac{\delta H}{\delta v} = \frac{\partial H}{\partial v} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial H}{\partial v_x} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial H}{\partial v_{xx}} + \dots$$

is the variational derivative.

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is the variational derivative.

Actually, we are dealing with integrals:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f(x, v, v_x, \dots)\mathrm{d}x = \int \frac{\delta f}{\delta v}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta H}{\delta v}\mathrm{d}x$$

The corresponding Poisson bracket is

$$\{\textstyle\int f\mathrm{d}x, \int g\mathrm{d}x\} = \displaystyle\int \frac{\delta f}{\delta v}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta g}{\delta v}\mathrm{d}x$$

(Skew symmetric due to integration by parts.)

Bi-Hamiltonian structure of the KdV hierarchy

We found one Poisson bracket:

$$\{\int f \mathrm{d}x, \int g \mathrm{d}x\}_1 = \int \frac{\delta f}{\delta v} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\delta g}{\delta v} \mathrm{d}x$$

There is also a second Poisson structure $\{\int f \mathrm{d}x, \int g \mathrm{d}x\}_2$, and Hamilton functions $H_1, H_3, H_5, \ldots$, such that

$$\begin{aligned}
\int v_{t_3} \mathrm{d}x &= \{\int v \mathrm{d}x, \int H_3 \mathrm{d}x\}_1 = \{\int v \mathrm{d}x, \int H_1 \mathrm{d}x\}_2 \\
\int v_{t_5} \mathrm{d}x &= \{\int v \mathrm{d}x, \int H_5 \mathrm{d}x\}_1 = \{\int v \mathrm{d}x, \int H_3 \mathrm{d}x\}_2 \\
\int v_{t_7} \mathrm{d}x &= \{\int v \mathrm{d}x, \int H_7 \mathrm{d}x\}_1 = \{\int v \mathrm{d}x, \int H_5 \mathrm{d}x\}_2 \\
&\vdots
\end{aligned}$$

This gives us another way to (recursively) construct the KdV hierarchy.

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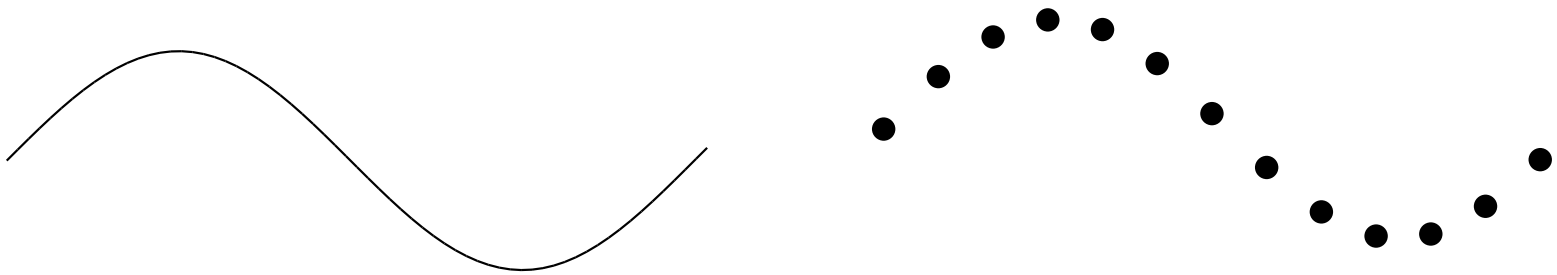
The problem of integrable discretization

- Many notions of integrability have a discrete counterpart
  $\rightarrow$ Integrable difference equations.
- Numerical discretizations almost always destroy integrability.
- What is the link between the continuous and discrete worlds?



Quad equations

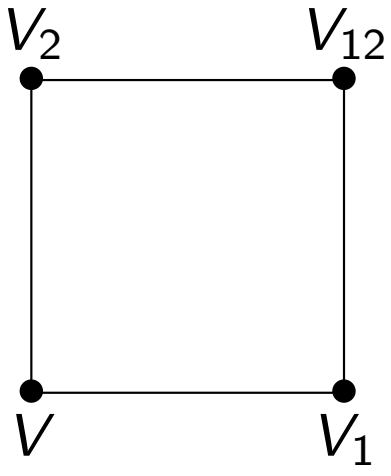
$$Q(V, V_1, V_2, V_{12}, \alpha_1, \alpha_2) = 0 \quad \text{on } \mathbb{Z}^2$$

Subscripts of V denote lattice shifts,

α_1, α_2 are parameters.

Invariant under symmetries of the square.

Affine in each of V, V_1, V_2, V_{12}.



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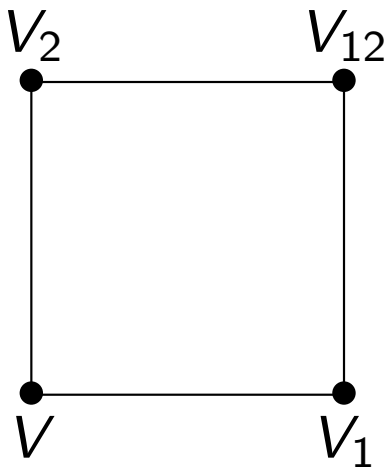
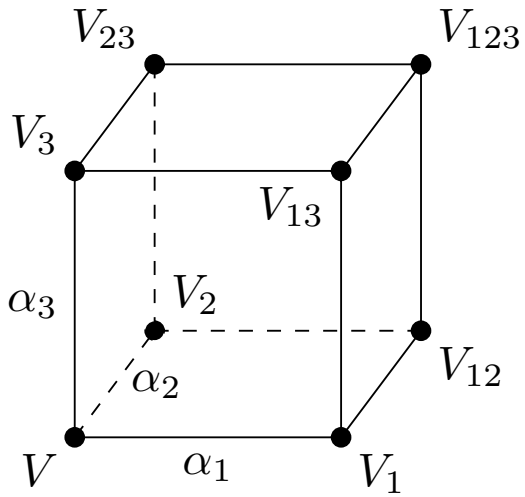


Integrability for systems quad equations: multi-dimensional consistency of

$$Q(V, V_i, V_j, V_{ij}, \alpha_i, \alpha_j) = 0,$$

i.e. given V, V_1, V_2 and V_3, the three ways of calculating V_{123} give the same result.



Example: discrete KdV equation

$$(V - V_{12})(V_1 - V_2) - \alpha_1 + \alpha_2 = 0$$

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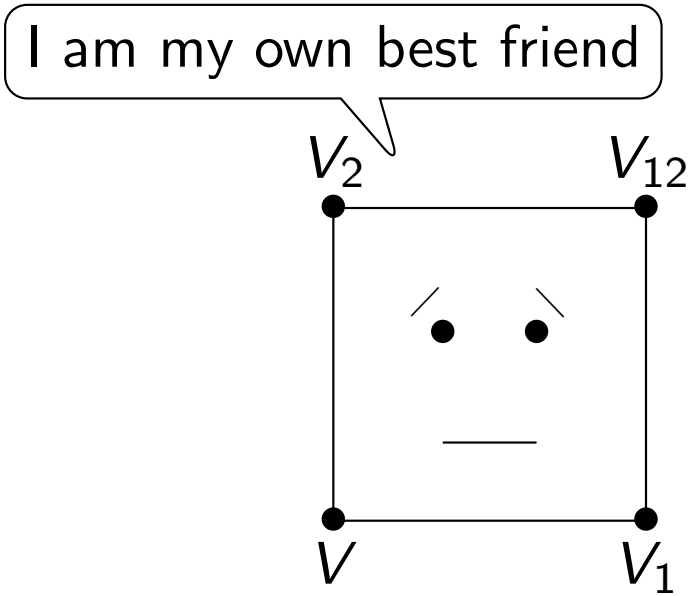

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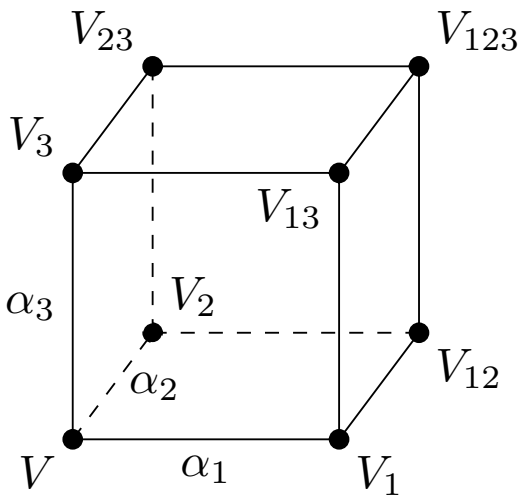

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Integrable continuum limits

Consider a continuous field $v : \mathbb{R}^n \to \mathbb{R}$, depending on time variables $(x = t_1, t_2, \ldots, t_n)$

Identify each lattice shift with some shift in $\mathbb{R}^n$, depending on α_1, α_2:

$$\begin{aligned}
V &= v(x, & t_2, & \ldots, t_n) \\
V_1 &= v(x + \alpha_1, & t_2 + \alpha_1^2, & \ldots, t_n + \alpha_1^n) \\
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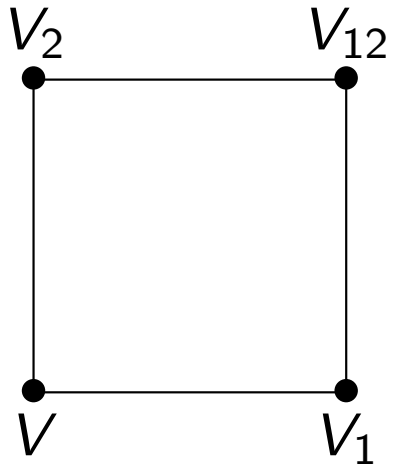
Then we can write $Q(V, V_1, V_2, V_{12}, \alpha_1, \alpha_2) = 0$ as a power series:

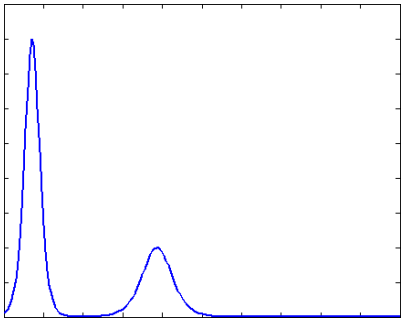
$$\sum_{i,j \in \mathbb{N}} F_{i,j}[v]\, \alpha_1^i \alpha_2^j = 0.$$



Example: for the discrete KdV equation, we find that $F_{0,j} = 0$ $(j = 3, 5, \ldots)$ is the KdV hierarchy.

Summary

Integrable systems pretend to be linear.



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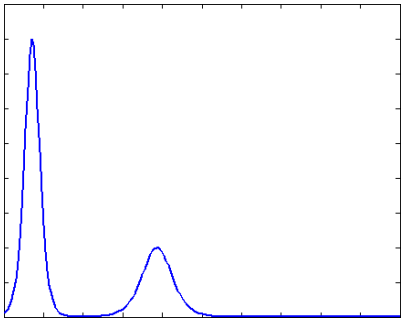

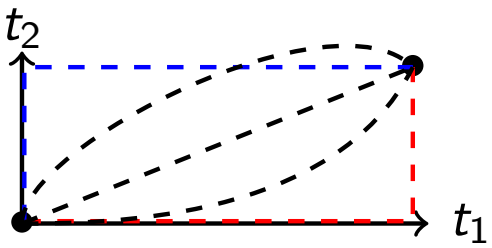

They can do so because they are part of a hierarchy of compatible equations $\Rightarrow$ many conserved quantities.

Tools to describe such hierarchies: Hamiltonian structures, Lax pairs, ...

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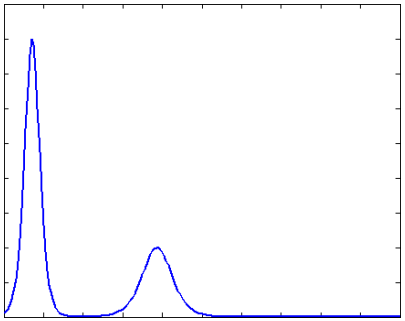

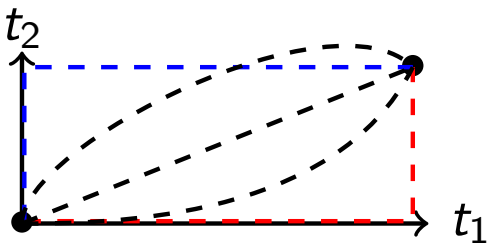

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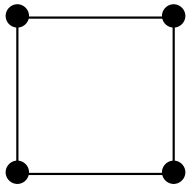
A single integrable difference equation can generate a full family of integrable differential equations.



Suggested reading

O. Babelon, D. Bernard, M. Talon. Introduction to classical integrable systems. Cambridge University Press, 2003.

A. Kasman. Glimpses of Soliton Theory: The Algebra and Geometry of Nonlinear PDEs. AMS, 2010.

Yu. Suris. The problem of integrable discretization: Hamiltonian approach. Birkhäuser, 2012.

J. Hietarinta, N. Joshi, F. Nijhoff. Discrete Systems and Integrability. Cambridge University Press, 2016.

Thank you for your attention!