Continuum limits of integrable lattice systems and their variational structure

Mats Vermeeren

Technische Universität Berlin

November 3, 2017

Integrable Systems Seminar, University of Leeds

Discretization in Geometry and Dynamics SFB Transregio 109

Berlin Mathematical School

Contents

1. Pluri-Lagrangian systems / Lagrangian multiform systems
   - $d = 2$, discrete
   - $d = 2$, continuous

$\searrow$

A variational structure for multi-dimensionally consistent equations.

	continuous	discrete
$d = 1$	ODEs	maps
$d \geq 2$	PDEs	PΔEs

2. Continuum limits
   - Proof by example: H1
   - Other equations

1. Pluri-Lagrangian systems / Lagrangian multiform systems
   - $d = 2$, discrete
   - $d = 2$, continuous
2. Continuum limits
   - Proof by example: H1
   - Other equations

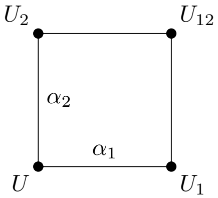
Quad equations

Quad equation on $\mathbb{Z}^2$:

$$Q(U, U_1, U_2, U_{12}, \alpha_1, \alpha_2) = 0$$

Subscripts of U denote lattice shifts, α_1, α_2 are parameters.

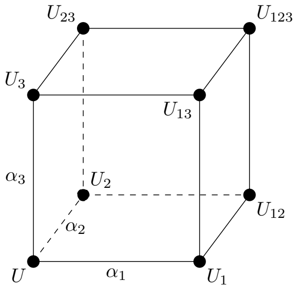
Q invariant under symmetries of the square, affine in each of U, U_1, U_2, U_{12}.



Integrability for systems of quad eqns: Multi-dimensional consistency of

$$Q(U, U_i, U_j, U_{ij}, \alpha_i, \alpha_j) = 0,$$

i.e. given U, U_1, U_2, U_3, the three ways of calculating U_{123} give the same result.



Quad equations

- Classification multidimensionally consistent quad equations in the ABS list.

  [Adler, Bobenko, Suris. Classification of integrable equations on quad-graphs. The consistency approach. Commun. Math. Phys. 2003.]

- Variational formulation in which the Lagrangian is a higher-dimensional object, encoding a full system of multidimensionally consistent equations.

  [Lobb, Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009.]

Pluri-Lagrangian principle ($d = 2$, discrete)

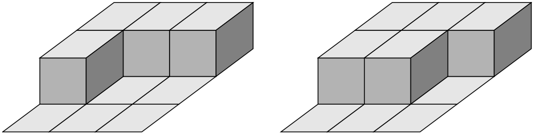
For some discrete 2-form

$$L(\sigma_{ij}) = L(U, U_i, U_j, U_{ij}, \alpha_i, \alpha_j),$$

find a field $U : \mathbb{Z}^N \to \mathbb{C}$ such that the action

$$\textstyle\sum_{\sigma_{ij} \in S} L(\sigma_{ij})$$

is critical on all discrete 2-surfaces S in $\mathbb{Z}^N$ simultaneously.



- EL equations obtained from corners of cubes.
- All ABS equations can be described this way.

Example: H1

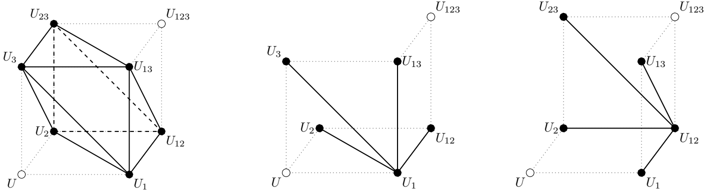
For the discrete potential KdV equation (H1)

$$(U - U_{ij})(U_i - U_j) - \alpha_i + \alpha_j = 0$$

we have the Lagrangian

$$L(U, U_i, U_j, U_{ij}, \alpha_i, \alpha_j) = UU_i - UU_j - (\alpha_i - \alpha_i)\log(U_i - U_j)$$

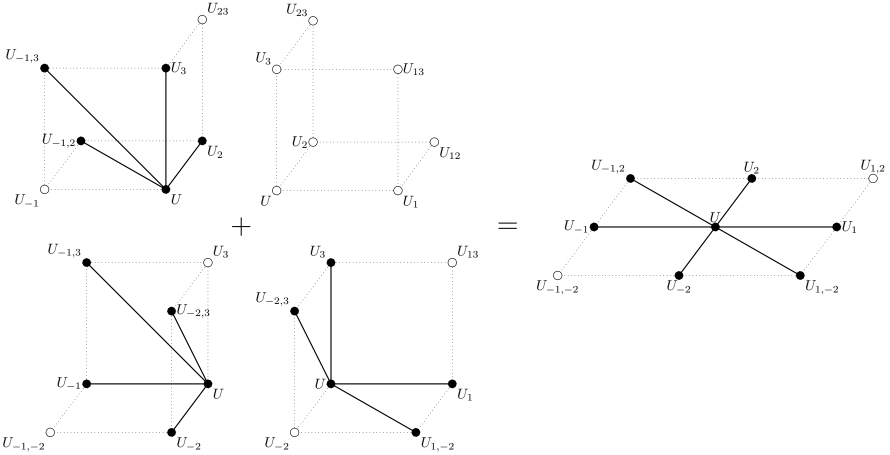
To derive Euler-Lagrange equations, we look at corners of a cube:



(a) (b)

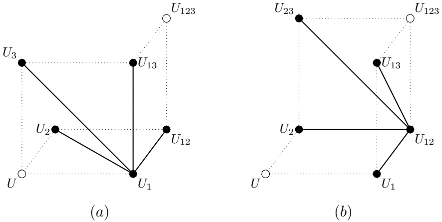
Example: H1

The conditions for other shapes follow from these, for example:



$\rightarrow$ corners of cubes are elementary building blocks for all discrete surfaces.

Example: H1



Depending on the orientation, we get the Euler-Lagrange equations

$$(a) \quad U_{ij} - U_{ik} - \frac{\alpha_i - \alpha_k}{U_i - U_k} + \frac{\alpha_i - \alpha_j}{U_i - U_j} = 0,$$

$$(b) \quad U_i - U_j - \frac{\alpha_j - \alpha_k}{U_{ij} - U_{ik}} + \frac{\alpha_i - \alpha_k}{U_{ij} - U_{jk}} = 0.$$

They are consequences of, but not equivalent to, the quad equation

$$(U - U_{ij})(U_i - U_j) - \alpha_i + \alpha_j = 0.$$

1. Pluri-Lagrangian systems / Lagrangian multiform systems
   - $d = 2$, discrete
   - $d = 2$, continuous

2. Continuum limits
   - Proof by example: H1
   - Other equations

Background: Lagrangian PDEs

Lagrangian density $\mathcal{L}(v, v_t, v_x, v_{tt}, v_{xt}, v_{xx}, \ldots)$

Action $\mathcal{S} = \int \mathcal{L} \, \mathrm{d}x \, \mathrm{d}t$

Look for a function v that is a critical point of the action, i.e. for arbitrary infinitesimal variations δv:

$$
\begin{aligned}
0 = \delta \mathcal{S} = \int \delta \mathcal{L} \, \mathrm{d}x \, \mathrm{d}t &= \int \sum_I \frac{\partial \mathcal{L}}{\partial v_I} \delta v_I \, \mathrm{d}x \, \mathrm{d}t \\
&= \int \sum_I (-1)^{|I|} \left(\mathrm{D}_I \frac{\partial \mathcal{L}}{\partial v_I} \right) \delta v \, \mathrm{d}x \, \mathrm{d}t,
\end{aligned}
$$

where for $I = (i_1, \ldots, i_k)$: $\quad \mathrm{D}_I = \dfrac{\mathrm{d}^{i_1}}{\mathrm{d}^{i_1} t_1} \cdots \dfrac{\mathrm{d}^{i_k}}{\mathrm{d}^{i_k} t_k} \quad$ and $\quad v_I = \mathrm{D}_I v$.

Euler-Lagrange equation:

$$
\frac{\delta \mathcal{L}}{\delta v} := \sum_I (-1)^{|I|} \mathrm{D}_I \frac{\partial \mathcal{L}}{\partial v_I} = 0
$$

KdV equation, $\mathcal{L} = \frac{1}{2} v_x v_t - v_x^3 - \frac{1}{2} v_x v_{xxx}$

Euler-Lagrange equation:

$$\begin{aligned}
0 = \frac{\delta \mathcal{L}}{\delta v} &= \sum_I (-1)^{|I|} \mathrm{D}_I \frac{\partial \mathcal{L}}{\partial v_I} \\
&= -\frac{1}{2} \mathrm{D}_t(v_x) - \frac{1}{2} \mathrm{D}_x(v_t) + 3 \mathrm{D}_x(v_x^2) + \frac{1}{2} \mathrm{D}_x(v_{xxx}) + \frac{1}{2} \mathrm{D}_{xxx}(v_x) \\
&= -v_{xt} + 6 v_x v_{xx} + v_{xxxx} \\
\Rightarrow v_{xt} &= 6 v_x v_{xx} + v_{xxxx}
\end{aligned}$$

Substitute $u = v_x$ to find the Korteweg-de Vries equation

$$u_t = 6 u u_x + u_{xxx}.$$

Or integrate to find the Potential Korteweg-de Vries equation

$$v_t = 3 v_x^2 + v_{xxx}.$$

Recall lattice potential KdV: Euler-Lagrange equation a consequence of, but not equivalent to, the quad equation.

What about integrability?

Integrable systems like the Toda lattice and the KdV equation come with infinite hierarchies. Each individual equation is Lagrangian/Hamiltonian.

On the Hamiltonian side it is clear when the equations of a hierarchy fit together:

$$\{H_i, H_j\} = 0.$$

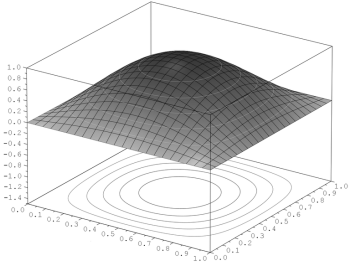
Arguably, the variational analogue of this property is a pluri-Lagrangian structure

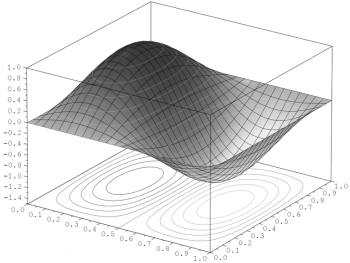
Pluri-Lagrangian principle ($d = 2$, continuous)

Given a 2-form

$$\mathcal{L} = \sum_{i,j} \mathcal{L}_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $u : \mathbb{R}^N \to \mathbb{C}$, such that $\displaystyle\int_\Gamma \mathcal{L}$ is critical on all smooth 2-surfaces Γ in multi-time $\mathbb{R}^N$.





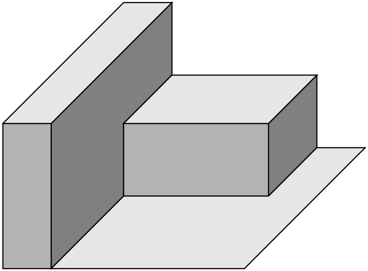
How to calculate Euler-Lagrange equations? Unlike the discrete case there are no elementary building blocks of smooth surfaces.

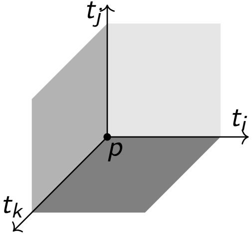
Multi-time EL equations

Consider a Lagrangian 2-form $\mathcal{L} = \sum_{i,j} \mathcal{L}_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$.

Every smooth surface can be approximated arbitrarily well by stepped surfaces.

Hence it is sufficient to require criticality on stepped surfaces, or just on their elementary corners.



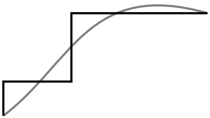


The case $d = 1$

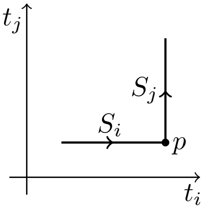
Consider a Lagrangian one-form $\mathcal{L} = \sum_i \mathcal{L}_i[u] \, \mathrm{d}t_i$

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.

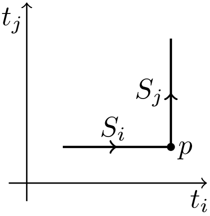


Variations are local, so it is sufficient to look at a general L-shaped curve $S = S_i \cup S_j$.



Multi-time Euler-Lagrange equations for $d = 1$

The variation of the action on S_i is



$$\delta \int_{S_i} \mathcal{L}_i \,\mathrm{d}t_i = \int_{S_i} \sum_I \frac{\partial \mathcal{L}_i}{\partial u_I} \delta u_I \,\mathrm{d}t_i$$
$$= \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i \mathcal{L}_i}{\delta u_I} \delta u_I \,\mathrm{d}t_i + \sum_I \frac{\delta_i \mathcal{L}_i}{\delta u_{It_i}} \delta u_I \bigg|_p ,$$

where I denotes a multi-index, and

$$\frac{\delta_i \mathcal{L}_i}{\delta u_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial \mathcal{L}_i}{\partial u_{It_i^\alpha}} = \frac{\partial \mathcal{L}_i}{\partial u_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial \mathcal{L}_i}{\partial u_{It_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial \mathcal{L}_i}{\partial u_{It_i^2}} - \ldots$$

Multi-time Euler-Lagrange equations for $\mathcal{L} = \sum_i \mathcal{L}_i[u] \,\mathrm{d}t_i$:

$$\frac{\delta_i \mathcal{L}_i}{\delta u_I} = 0 \qquad \forall I \not\ni t_i \qquad \text{and} \qquad \frac{\delta_i \mathcal{L}_i}{\delta u_{It_i}} = \frac{\delta_j \mathcal{L}_j}{\delta u_{It_j}} \qquad \forall I,$$

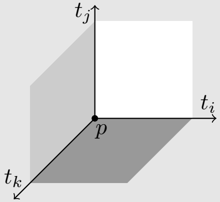
Multi-time EL equations for $d = 2$

For $\mathcal{L} = \sum_{i,j} \mathcal{L}_{ij}[u] \,\mathrm{d}t_i \wedge \mathrm{d}t_j$:

$$\frac{\delta_{ij}\mathcal{L}_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij}\mathcal{L}_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik}\mathcal{L}_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij}\mathcal{L}_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk}\mathcal{L}_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki}\mathcal{L}_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij}\mathcal{L}_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}(-1)^{\alpha+\beta}\frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha}\frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta}\frac{\partial \mathcal{L}_{ij}}{\partial u_{It_i^\alpha t_j^\beta}}$$

[Suris, V. On the Lagrangian structure of integrable hierarchies. In AI Bobenko (ed): Advances in Discrete Differential Geometry, Springer. 2016.]

Example: Potential KdV hierarchy

$$v_{t_2} = g_2[v] = v_{xxx} + 3v_x^2,$$

$$v_{t_3} = g_3[v] = v_{xxxxx} + 10v_x v_{xxx} + 5v_{xx}^2 + 10v_x^3,$$

where we identify $t_1 = x$.

The differentiated equations $v_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x} g_i[v]$ are Lagrangian with

$$\mathcal{L}_{12} = \frac{1}{2} v_x v_{t_2} - \frac{1}{2} v_x v_{xxx} - v_x^3,$$

$$\mathcal{L}_{13} = \frac{1}{2} v_x v_{t_3} - v_x v_{xxxxx} - 2v_{xx} v_{xxxx} - \frac{3}{2} v_{xxx}^2 + 5v_x^2 v_{xxx} + 5v_x v_{xx}^2 + \frac{5}{2} v_x^4.$$

We choose the coefficient $\mathcal{L}_{23}$ of

$$\mathcal{L} = \mathcal{L}_{12}[u] \,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + \mathcal{L}_{13}[u] \,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + \mathcal{L}_{23}[u] \,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

such that the pluri-Lagrangian 2-form is closed on solutions. It is of the form

$$\mathcal{L}_{23} = \frac{1}{2}\big(v_{t_2} g_3[v] - v_{t_3} g_2[v]\big) + p_{23}[v].$$

Example: Potential KdV hierarchy

- The equations

$$\frac{\delta_{12}\mathcal{L}_{12}}{\delta v} = 0 \qquad \text{and} \qquad \frac{\delta_{13}\mathcal{L}_{13}}{\delta v} = 0$$

are

$$v_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x} g_2[v] \qquad \text{and} \qquad v_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x} g_3[v].$$

- The equations

$$\frac{\delta_{12}\mathcal{L}_{12}}{\delta v_x} = \frac{\delta_{32}\mathcal{L}_{32}}{\delta v_{t_3}} \qquad \text{and} \qquad \frac{\delta_{13}\mathcal{L}_{13}}{\delta v_x} = \frac{\delta_{23}\mathcal{L}_{23}}{\delta v_{t_2}}$$

yield

$$v_{t_2} = g_2 \qquad \text{and} \qquad v_{t_3} = g_3,$$

the evolutionary equations!

- All other multi-time EL equations are corollaries of these.

1. Pluri-Lagrangian systems / Lagrangian multiform systems
   - $d = 2$, discrete
   - $d = 2$, continuous

2. Continuum limits
   - Proof by example: H1
   - Other equations

Continuum limit of H1 (lattice potential KdV)

We make a non-autonomous change of variables to transform $(X - X_{12})(X_2 - X_1) = \alpha_1 - \alpha_2$ into

$$\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + U_{1,2} - U \right) \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} + U_2 - U_1 \right) = \frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2} \qquad \text{(lpKdV)}$$

Method by Wiersma and Capel produces the pKdV hierarchy from (lpKdV)

They used an differential-difference equation as intermediate step.
Here we will present the same limit in a single step.

[Wiersma, Capel. Lattice equations, hierarchies and Hamiltonian structures. Physica A. 1987]

Continuum limit of H1 (lattice potential KdV)

Miwa shifts

Skew embedding of the mesh $\mathbb{Z}^N$ into multi-time $\mathbb{R}^N$
Discrete $U : \mathbb{Z}^N \to \mathbb{C}$ is a sampling of the continuous $u : \mathbb{R}^N \to \mathbb{C}$:

$$U = U(\mathbf{n}) = u(t_1, t_2, \ldots, t_N),$$

$$U_i = U(\mathbf{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \ldots, t_N + 2(-1)^N \frac{\lambda_i^N}{N} \right)$$

[Miwa. On Hirota's difference equations. Proceedings of the Japan Academy A. 1982]

Plug into $\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + U_{1,2} - U \right) \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} + U_2 - U_1 \right) = \frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2}$
and expand in λ_1, λ_2.

In leading order everything cancels (due to very specific form of quad eq).
$\hookrightarrow$ generically we would have an ODE in t_1.

Continuum limit of H1 (lattice potential KdV)

Series expansion

$$\text{Quad Equation} \quad \rightarrow \quad \sum_{i,j} (-1)^{i+j} \frac{4}{ij} \, f_{i,j}[u] \, \lambda_1^i \lambda_2^j = 0,$$

where $f_{j,i} = -f_{i,j}$ and the factor $(-1)^{i+j}\dfrac{4}{ij}$ is chosen to normalize the $f_{0,j}$.

First row of coefficients:

$$\begin{aligned}
f_{0,1} &= u_{t_2}, \\
f_{0,2} &= -3u_{t_1}^2 - u_{t_1 t_1 t_1} - \frac{3}{2} u_{t_1 t_2} + u_{t_3}, \\
f_{0,3} &= -8 u_{t_1} u_{t_1 t_1} - 4 u_{t_1} u_{t_2} - \frac{4}{3} u_{t_1 t_1 t_1 t_1} + \frac{4}{3} u_{t_1 t_3} + u_{t_2 t_2} + u_{t_4}, \\
f_{0,4} &= -5 u_{t_1 t_1}^2 - \frac{20}{3} u_{t_1} u_{t_1 t_1 t_1} + 10 u_{t_1} u_{t_1 t_2} + 5 u_{t_1 t_1} u_{t_2} - \frac{5}{4} u_{t_2}^2 - \frac{10}{3} u_{t_1} u_{t_3} \\
&\qquad - u_{t_1 t_1 t_1 t_1 t_1} + \frac{5}{3} u_{t_1 t_1 t_1 t_2} + \frac{5}{4} u_{t_1 t_2 t_2} - \frac{5}{4} u_{t_1 t_4} - \frac{5}{3} u_{t_2 t_3} + u_{t_5}, \\
&\ \ \vdots
\end{aligned}$$

Continuum limit of H1 (lattice potential KdV)

Setting each f_{ij} equal to zero, we find

$$
\begin{aligned}
u_{t_2} &= 0, \\
u_{t_3} &= 3u_{t_1}^2 + u_{t_1 t_1 t_1} \\
u_{t_4} &= 0, \\
u_{t_5} &= 10u_{t_1}^3 + 5u_{t_1 t_1}^2 + 10u_{t_1} u_{t_1 t_1 t_1} + u_{t_1 t_1 t_1 t_1 t_1}, \\
&\ \ \vdots
\end{aligned}
$$

$\hookrightarrow$ pKdV hierarchy

Whole hierarchy from single quad equation

using Miwa correspondence

$$
\begin{aligned}
U &= U(\mathbf{n}) = u(t_1, t_2, \ldots, t_N), \\
U_i &= U(\mathbf{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \ldots, t_N + 2(-1)^N \frac{\lambda_i^N}{N} \right)
\end{aligned}
$$

Continuum limit of the Lagrangian for H1

A Lagrangian for (lpKdV) is

$$L(\square) = \frac{1}{2}\left(U - U_{i,j} - \lambda_i^{-1} - \lambda_j^{-1}\right)\left(U_i - U_j + \lambda_i^{-1} - \lambda_j^{-1}\right) \\ + \left(\lambda_i^{-2} - \lambda_j^{-2}\right)\log\left(1 + \frac{U_i - U_j}{\lambda_i^{-1} - \lambda_j^{-1}}\right).$$

A particular (non-standard) choice among many equivalent Lagrangians.

$\hookrightarrow$ to get leading-order cancellation.

Using Miwa correspondence:

$$\text{Discrete } L \quad \rightarrow \quad \text{Power series } \mathcal{L}_{\mathrm{disc}}[u(\mathbf{t})]$$

where $[\cdot]$ denotes dependence on the infinite jet bundle. If $\square \mapsto \blacksquare$:

$$L(\square) = \mathcal{L}_{\mathrm{disc}}[u(\text{corner of } \blacksquare)].$$

Continuum limit of the Lagrangian

Series expansion is not the end of the story. The action would still be a sum:

$$S = \sum_{\square \in \sigma} L(\square) \;=\; \sum_{\blacksquare \in \text{ embedded } \sigma} \mathcal{L}_{\mathrm{disc}}[u(\text{corner of } \blacksquare)].$$

We want an integral: $\quad S = \displaystyle\int_\sigma \mathcal{L}. \qquad$ Locally: $\quad L(\square) = \displaystyle\int_{\blacksquare} \mathcal{L}.$

Euler-MacLaurin formula

$$\begin{aligned}
\sum_{j=0}^{n-1} g(j) &= \int_0^n g(t)\,\mathrm{d}t + \sum_{i=1}^{\infty} \frac{B_i}{i!}\left(g^{(i-1)}(n) - g^{(i-1)}(0)\right) \\
&= \int_0^n \left(g(t) + \sum_{i=1}^{\infty} \frac{B_i}{i!} g^{(i)}(t)\right)\mathrm{d}t,
\end{aligned}$$

where the B_i are the Bernoulli numbers $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \cdots$.

Continuum limit of the Lagrangian

Two applications of the EM formula (in Miwa coordinates):

$$\mathcal{L}_{\mathrm{Miwa}}([u],\lambda_1,\lambda_2) = \sum_{i,j=0}^{\infty} \frac{B_i B_j}{i!j!} \partial_1^i \partial_2^j \mathcal{L}_{\mathrm{disc}}([u],\lambda_1,\lambda_2).$$

where the differential operators are $\quad \partial_k = \displaystyle\sum_{j=1}^{N} (-1)^{j+1} \frac{2\lambda_k^j}{j} \frac{\mathrm{d}}{\mathrm{d}t_j}$.

Then there holds

$$L_{\mathrm{disc}}(\square) = \int_{\blacksquare} \mathcal{L}_{\mathrm{Miwa}}([u(\mathbf{t})],\lambda_1,\lambda_2)\, \eta_1 \wedge \eta_2,$$

where η_1 and η_2 are the one-forms dual to the Miwa shifts.

This suggests the Lagrangian 2-form

$$\sum_{1 \leq i < j \leq N} \mathcal{L}_{\mathrm{Miwa}}([u],\lambda_i,\lambda_j)\, \eta_i \wedge \eta_j.$$

Continuum limit of the Lagrangian

Lemma

If every term in the power series $\mathcal{L}_{\mathrm{Miwa}}$ is of degree > 0 in each λ_i,

$$\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=1}^{\infty} (-1)^{i+j} 4 \frac{\lambda_1^i}{i} \frac{\lambda_2^j}{j} \mathcal{L}_{i,j}[u],$$

then

$$\sum_{1 \leq i < j \leq N} \mathcal{T}_N\big(\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_i, \lambda_j)\big)\, \eta_i \wedge \eta_j = \sum_{1 \leq i < j \leq N} \mathcal{L}_{i,j}[u]\, \mathrm{d}t_i \wedge \mathrm{d}t_j \quad =: \mathcal{L}.$$

Proof Evaluate $\mathcal{L}$ on the Miwa shifts $(\mathfrak{v}_k, \mathfrak{v}_l)$ dual to (η_k, η_l):

$$\big\langle \mathcal{L}, (\mathfrak{v}_k, \mathfrak{v}_l) \big\rangle = \Bigg\langle \sum_{1 \leq i < j \leq N} \mathcal{L}_{ij}[u]\, \mathrm{d}t_i \wedge \mathrm{d}t_j \;,\; (\mathfrak{v}_k, \mathfrak{v}_l) \Bigg\rangle$$

$$= \sum_{i,j=1}^{N} \mathcal{L}_{ij}[u] (-1)^{i+j} 4 \frac{\lambda_k^i}{i} \frac{\lambda_l^j}{j} \quad = \mathcal{T}_N\big(\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_k, \lambda_l)\big) \quad \square$$

Continuum limit of the Lagrangian

Theorem

If $\quad \mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2) = \displaystyle\sum_{i,j=1}^{\infty} (-1)^{i+j} 4 \frac{\lambda_1^i}{i} \frac{\lambda_2^j}{j} \mathcal{L}_{ij}[u],$

then $\quad \mathcal{L} = \displaystyle\sum_{1 \leq i < j \leq N} \mathcal{L}_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j \quad$ is a pluri-Lagrangian structure.

Proof

$$L_{\mathrm{disc}}(\square) = \int_{\blacksquare} \sum_{1 \leq i < j \leq N} \mathcal{L}_{\mathrm{Miwa}}([u(\mathbf{t})], \lambda_i, \lambda_j) \, \eta_i \wedge \eta_j$$

$$= \int_{\blacksquare} \sum_{1 \leq i < j \leq N} \mathcal{L}_{i,j}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j \quad + \mathcal{O}(\lambda_1^{N+1} + \ldots + \lambda_N^{N+1}).$$

So for $\Gamma_{\mathrm{disc}} \mapsto \Gamma$ under the Miwa correspondence:

$$\sum_{\Gamma_{\mathrm{disc}}} L(\square) = \int_{\Gamma} \mathcal{L} \quad + \mathcal{O}(\lambda_1^{N+1} + \ldots + \lambda_N^{N+1}).$$

Such Γ are sufficient to derive all multi-time EL equations. $\square$

Summary of the algorithm

$L(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2)$ Suitable choice $\Rightarrow$ leading order cancellation

$$\Big\downarrow \text{ Miwa shifts, Taylor expansion}$$

$$\mathcal{L}_{\mathrm{disc}}([u], \lambda_1, \lambda_2)$$

$$\Big\downarrow \text{ Euler-Maclaurin formula}$$

$$\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=1}^{\infty} (-1)^{i+j} 4 \frac{\lambda_1^i}{i} \frac{\lambda_2^j}{j} \mathcal{L}_{i,j}[u]$$

$$\Big\downarrow \qquad\qquad \Big\downarrow$$

$$\sum_{1 \leq i < j \leq N} \mathcal{L}_{\mathrm{Miwa}}([u], \lambda_i, \lambda_j)\, \eta_i \wedge \eta_j \approx \sum_{1 \leq i < j \leq N} \mathcal{L}_{i,j}[u]\, \mathrm{d}t_i \wedge \mathrm{d}t_j$$

Continuum limit of the Lagrangian for H1

The first row of coefficients is (with $u_k = u_{t_k}$)

$$\begin{aligned}
\mathcal{L}_{12} &= u_1 u_2 \\
\mathcal{L}_{13} &= -2u_1^3 - u_1 u_{111} + \frac{3}{4} u_2^2 + u_1 u_3 \\
\mathcal{L}_{14} &= -4u_1^2 u_2 - \frac{4}{3} u_1 u_{112} - \frac{2}{3} u_{11} u_{12} - \frac{2}{3} u_{111} u_2 + \frac{4}{3} u_2 u_3 + u_1 u_4 \\
\mathcal{L}_{15} &= \frac{10}{3} u_1 u_{11}^2 - \frac{5}{2} u_1 u_2^2 - \frac{10}{3} u_1^2 u_3 + \frac{5}{9} u_{11} u_{1111} + \frac{1}{9} u_1 u_{11111} \\
&\qquad - \frac{10}{9} u_1 u_{113} - \frac{5}{6} u_{12}^2 - \frac{5}{12} u_1 u_{122} - \frac{5}{9} u_{11} u_{13} - \frac{5}{6} u_{112} u_2 \\
&\qquad - \frac{5}{12} u_{11} u_{22} - \frac{5}{9} u_{111} u_3 + \frac{5}{9} u_3^2 + \frac{5}{4} u_2 u_4 + u_1 u_5 \\
&\ \ \vdots
\end{aligned}$$

Continuum limit of the Lagrangian for H1

The first row of coefficients is (with $u_k = u_{t_k}$)

$$\mathcal{L}_{12} = u_1 u_2$$

$$\mathcal{L}_{13} = -2u_1^3 - u_1 u_{111} + \frac{3}{4} u_2^2 + u_1 u_3$$

$$\mathcal{L}_{14} = -4u_1^2 u_2 - \frac{4}{3} u_1 u_{112} - \frac{2}{3} u_{11} u_{12} - \frac{2}{3} u_{111} u_2 + \frac{4}{3} u_2 u_3 + u_1 u_4$$

$$\begin{aligned}\mathcal{L}_{15} = {} & \frac{10}{3} u_1 u_{11}^2 - \frac{5}{2} u_1 u_2^2 - \frac{10}{3} u_1^2 u_3 + \frac{5}{9} u_{11} u_{1111} + \frac{1}{9} u_1 u_{11111} \\ & - \frac{10}{9} u_1 u_{113} - \frac{5}{6} u_{12}^2 - \frac{5}{12} u_1 u_{122} - \frac{5}{9} u_{11} u_{13} - \frac{5}{6} u_{112} u_2 \\ & - \frac{5}{12} u_{11} u_{22} - \frac{5}{9} u_{111} u_3 + \frac{5}{9} u_3^2 + \frac{5}{4} u_2 u_4 + u_1 u_5\end{aligned}$$

$$\vdots$$

How to get rid of these alien derivatives?

Eliminating alien derivatives

A derivative u_{t_k} or u_{It_k} with $k \neq 1, i, j$ is called alien to $\mathcal{L}_{i,j}$

We can (in this case) get rid of the alien derivatives in each $\mathcal{L}_{1j}$ by

- adding a total derivative $\mathrm{D}_{t_1} c_j$, and
- discarding terms that have a double zero on solutions.

Then, to get an equivalent Lagrangian 2-form:

- also add $\mathrm{D}_{t_i} c_j$ to the coefficients $\mathcal{L}_{ij}$.

  This amounts to adding the exact form $\mathrm{d}\left(\sum_j c_j \mathrm{d}t_j\right)$ to $\mathcal{L}$.

The alien derivatives that are left in $\mathcal{L}_{ij}$ can now be eliminated naively:

Theorem

If the coefficients $\mathcal{L}_{1j}$ do not contain alien terms, then eliminating the alien derivatives in $\mathcal{L}_{ij}$ using the equations of motion yields an equivalent Lagrangian 2-form.

Continuum limit of the Lagrangian for H1

$\mathcal{L}_{1,2} = \frac{1}{2} u_1 u_2$ $\qquad$ $\mathcal{L}_{1,3} = -u_1^3 + \frac{1}{2} u_{11}^2 + \frac{1}{2} u_1 u_3$

$\mathcal{L}_{1,4} = \frac{1}{2} u_1 u_4$ $\qquad$ $\mathcal{L}_{1,5} = -\frac{5}{2} u_1^4 + 5 u_1 u_{11}^2 - \frac{1}{2} u_{111}^2 + \frac{1}{2} u_1 u_5$

$$\mathcal{L}_{2,3} = -3u_1^2 u_2 + u_{11} u_{12} - u_{111} u_2 + \frac{1}{2} u_2 u_3$$

$$\mathcal{L}_{2,4} = \frac{1}{2} u_2 u_4$$

$$\begin{aligned}\mathcal{L}_{2,5} = &-10 u_1^3 u_2 + 10 u_1 u_{11} u_{12} - 5 u_{11}^2 u_2 - 10 u_1 u_{111} u_2 - u_{111} u_{112} + \\ &u_{1111} u_{12} - u_{11111} u_2 + \frac{1}{2} u_2 u_5\end{aligned}$$

$$\mathcal{L}_{3,4} = -u_{11} u_{14} + \frac{1}{2} u_3 u_4$$

$$\begin{aligned}\mathcal{L}_{3,5} = &\, 18 u_1^5 + 30 u_1^3 u_{111} - 10 u_1^3 u_3 + 6 u_{11}^2 u_{111} + 8 u_1 u_{111}^2 - 6 u_1 u_{11} u_{1111} + \\ &3 u_1^2 u_{11111} + 10 u_1 u_{11} u_{13} - 5 u_{11}^2 u_3 - 10 u_1 u_{111} u_3 - \frac{1}{2} u_{1111}^2 + \\ &u_{111} u_{11111} - u_{111} u_{113} + u_{1111} u_{13} - u_{11} u_{15} - u_{11111} u_3 + \frac{1}{2} u_3 u_5\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{4,5} = &-10 u_1^3 u_4 + 10 u_1 u_{11} u_{14} - 5 u_{11}^2 u_4 - 10 u_1 u_{111} u_4 - u_{111} u_{114} + \\ &u_{1111} u_{14} - u_{11111} u_4 + \frac{1}{2} u_4 u_5\end{aligned}$$

Continuum limit of the Lagrangian for H1

$\mathcal{L}_{1,2} = \frac{1}{2} u_1 u_2$

$\mathcal{L}_{1,3} = -u_1^3 + \frac{1}{2} u_{11}^2 + \frac{1}{2} u_1 u_3$

$\mathcal{L}_{1,4} = \frac{1}{2} u_1 u_4$

$\mathcal{L}_{1,5} = -\frac{5}{2} u_1^4 + 5 u_1 u_{11}^2 - \frac{1}{2} u_{111}^2 + \frac{1}{2} u_1 u_5$

$$\mathcal{L}_{2,3} = -3u_1^2 u_2 + u_{11} u_{12} - u_{111} u_2 + \frac{1}{2} u_2 u_3$$

$$\mathcal{L}_{2,4} = \frac{1}{2} u_2 u_4$$

$$\mathcal{L}_{2,5} = -10 u_1^3 u_2 + 10 u_1 u_{11} u_{12} - 5 u_{11}^2 u_2 - 10 u_1 u_{111} u_2 - u_{111} u_{112} + u_{1111} u_{12} - u_{11111} u_2 + \frac{1}{2} u_2 u_5$$

$$\mathcal{L}_{3,4} = -u_{11} u_{14} + \frac{1}{2} u_3 u_4$$

$$\mathcal{L}_{3,5} = 18 u_1^5 + 30 u_1^3 u_{111} - 10 u_1^3 u_3 + 6 u_{11}^2 u_{111} + 8 u_1 u_{111}^2 - 6 u_1 u_{11} u_{1111} + 3 u_1^2 u_{11111} + 10 u_1 u_{11} u_{13} - 5 u_{11}^2 u_3 - 10 u_1 u_{111} u_3 - \frac{1}{2} u_{1111}^2 + u_{111} u_{11111} - u_{111} u_{113} + u_{1111} u_{13} - u_{11} u_{15} - u_{11111} u_3 + \frac{1}{2} u_3 u_5$$

$$\mathcal{L}_{4,5} = -10 u_1^3 u_4 + 10 u_1 u_{11} u_{14} - 5 u_{11}^2 u_4 - 10 u_1 u_{111} u_4 - u_{111} u_{114} + u_{1111} u_{14} - u_{11111} u_4 + \frac{1}{2} u_4 u_5$$

1. Pluri-Lagrangian systems / Lagrangian multiform systems
    - $d = 2$, discrete
    - $d = 2$, continuous

2. Continuum limits
    - Proof by example: H1
    - Other equations

Other equations: $Q1_{\delta=0}$

$$\frac{U_2 - U}{\lambda_2}\frac{U_{1,2} - U_1}{\lambda_2} - \frac{U_1 - U}{\lambda_1}\frac{U_{1,2} - U_2}{\lambda_1} = 0$$

$$L = \lambda_1^2 \log\left(\frac{U - U_1}{\lambda_1}\right) - \lambda_2^2 \log\left(\frac{U - U_2}{\lambda_2}\right) - (\lambda_1^2 - \lambda_2^2)\log\left(\frac{U_2 - U_1}{\lambda_2 - \lambda_1}\right)$$

produces the Schwarzian KdV hierarchy

$$\frac{u_{t_2}}{u_{t_1}} = 0,$$

$$\frac{u_{t_3}}{u_{t_1}} = -\frac{3u_{t_1 t_1}^2}{2u_{t_1}^2} + \frac{u_{t_1 t_1 t_1}}{u_{t_1}},$$

$$\frac{u_{t_4}}{u_{t_1}} = 0,$$

$$\frac{u_5}{u_{t_1}} = -\frac{45u_{t_1 t_1}^4}{8u_{t_1}^4} + \frac{25u_{t_1 t_1}^2 u_{t_1 t_1 t_1}}{2u_{t_1}^3} - \frac{5u_{t_1 t_1 t_1}^2}{2u_{t_1}^2} - \frac{5u_{t_1 t_1} u_{t_1 t_1 t_1 t_1}}{u_{t_1}^2} + \frac{u_{t_1 t_1 t_1 t_1 t_1}}{u_{t_1}},$$

$$\vdots$$

Other equations: $Q1_{\delta=0}$

This continuum limit is implicitly contained in [Nijhoff, Hone, Joshi. On a Schwarzian PDE associated with the KdV hierarchy. 2000]

They consider a non-autonomous generating PDE for $z_{n,m}(t,s)$ depending on continuous variables (s,t), and parameters (m,n). It commutes with

$$\frac{(z_{n,m} - z_{n+1,m})(z_{n,m+1} - z_{n+1,m+1})}{(z_{n,m} - z_{n,m+1})(z_{n+1,m} - z_{n+1,m+1})} = \frac{s}{t} \qquad (Q1_{\delta=0})$$

The SKdV hierarchy is recovered through the identification

$$z_{n,m}(t,s) = v\left(x_1 + \frac{2n}{t^{\frac{1}{2}}} + \frac{2m}{s^{\frac{1}{2}}}, x_3 + \frac{2n}{3t^{\frac{3}{2}}} + \frac{2m}{3s^{\frac{3}{2}}}, \dots \right).$$

Setting $t = \lambda_1^{-2}$ and $s = \lambda_2^{-2}$ we obtain the odd order Miwa shifts.

$$\begin{tikzcd}
 & \boxed{\text{Generating PDE for } z_{n,m}(t,s)} \arrow[dl, "\text{role reversal } (n,m) \leftrightarrow (t,s)"'] \arrow[dr, "\text{Miwa expansion}"] & \\
\boxed{\text{Quad equation Q1}} \arrow[rr, "\text{continuum limit}"] & & \boxed{\text{SKdV hierarchy}}
\end{tikzcd}$$

Other equations: Q-list

All yield hierarchies starting with some version of the Krichever-Novikov equation

$$v_t = v_{xxx} - \frac{3}{2}\frac{v_{xx} - \frac{1}{4}}{v_x} - \frac{3}{2}\wp(v) v_x^3$$

$$\mathrm{Q1}_{\delta=1} \quad \to \quad v_3 = v_{111} - \frac{3}{2}\frac{v_{11}^2 - \frac{1}{4}}{v_1}$$

$$\mathrm{Q2} \quad \to \quad v_3 = v_{111} - \frac{3}{2}\frac{v_{11}^2 - \frac{1}{4}}{v_1} - \frac{3}{2}\frac{v_1^3}{v^2}$$

$$\mathrm{Q3}_{\delta=0} \quad \to \quad v_3 = v_{111} - \frac{3}{2}\frac{v_{11}^2 - \frac{1}{4}}{v_1} + \frac{1}{2} v_1^3$$

$$\mathrm{Q3}_{\delta=1} \quad \to \quad v_3 = v_{111} - \frac{3}{2}\frac{v_{11}^2 - \frac{1}{4}}{v_1} + \frac{1}{2} v_1^3 - \frac{3}{2}\frac{v_1^3}{\sin(v)^2}$$

$$\mathrm{Q4} \quad \to \quad v_3 = v_{111} - \frac{3}{2}\frac{v_{11}^2 - \frac{1}{4}}{v_1} + (1+k) v_1^3 + \frac{3}{2}\frac{(k-1)^2 \operatorname{sn}(v)^2}{\operatorname{cn}(v)^2 \operatorname{dn}(v)^2} v_1^3$$

($\hookrightarrow$ Lagrangian not explicitly known)

Other equations: $\mathrm{H3}_{\delta=0}$

Equation:

$$\lambda_1(e^{V+V_1} - e^{V_2+V_{1,2}}) - \lambda_2(e^{V+V_1} - e^{V_1+V_{1,2}}) = 0$$
$$\to v_3 = v_{111} - 2v_1^3, \ldots$$

But the Lagrangian

$$L = \frac{1}{2}\log\left(\frac{VV_1}{-\lambda_1}\right)^2 - \frac{1}{2}\log\left(\frac{VV_2}{-\lambda_2}\right)^2 + \mathrm{Li}_2\left(\frac{\lambda_2 V_1}{\lambda_1 U_2}\right) - \mathrm{Li}_2\left(\frac{\lambda_1 V_1}{\lambda_2 U_2}\right)$$
$$+ \log\left(\frac{\lambda_1^2}{\lambda_2^2}\right)\log(V) + \log(\lambda_2^2)\log\left(\frac{V_1}{V_2}\right)$$

does not allow a series expansion

Other equations: $H3_{\delta=0}$

$$\lambda_1 \sin\left(\frac{V_{12}+V_2-V_1-V}{4}\right) - \lambda_2 \sin\left(\frac{V_{12}-V_2+V_1-V}{4}\right) = 0.$$

Three leg form:

$$\arctan\left(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\tan\left(\frac{V_1-V_2}{4}\right)\right) = (V_{12}-V_1)-(V-V_1).$$

Lagrangian:

$$L = \frac{1}{2}(V_1-V)^2 - \frac{1}{2}(V_2-V)^2 - \mathcal{I}_{\lambda_1,\lambda_2}(V_1-V_2),$$

where

$$\mathcal{I}_{\lambda_1,\lambda_2}(x) = \int \arctan\left(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\tan\left(\frac{x}{4}\right)\right)\mathrm{d}x.$$

$\to$ potential modified KdV hierarchy $v_3 = v_{111} + \frac{1}{2}v_1^3, \ldots$

Other equations: Open problems

- H2 and H3$_{\delta \neq 0}$:

$$(U - U_{12})(U_1 - U_2) - (\lambda_1 - \lambda_2)(U + U_1 + U_2 + U_{12}) - \lambda_1^2 + \lambda_2^2 = 0$$

$$\lambda_1 (U U_1 + U_2 U_{1,2}) - \lambda_2 (U U_1 + U_1 U_{1,2}) + \delta(\lambda_1^2 - \lambda_2^2) = 0$$

Order mismatch in series expansion $\rightarrow$ no limit known.

- 3-dimensional case: lattice KP

$$\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_3} \right) U_1 U_{23} + \left(\frac{1}{\lambda_3} - \frac{1}{\lambda_1} \right) U_2 U_{13} + \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_1} \right) U_3 U_{12} = 0$$

$$\rightarrow v_{13} = \frac{3\, v_{11}^2}{v} - \frac{4\, v_1 v_{111}}{v} + v_{1111} - \frac{3\, v_2^2}{4\, v} + \frac{3}{4} v_{22} + \frac{v_1 v_3}{v}, \quad \ldots$$

No Lagrangian limit known.

Conclusion

Requirements:
- Multidimensional consistency.
- Suitable form of the equation and the Lagrangian.

Then everything is algorithmic:

$L(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2)$ Suitable form $\Rightarrow$ leading order cancellation

$\downarrow$ Miwa shifts, Taylor expansion

$\mathcal{L}_{\mathrm{disc}}([u], \lambda_1, \lambda_2)$

$\downarrow$ Euler-Maclaurin formula

$$\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=1}^{\infty} (-1)^{i+j} 4 \frac{\lambda_1^i}{i} \frac{\lambda_2^j}{j} \mathcal{L}_{i,j}[u]$$

$\downarrow$ $\downarrow$

$$\sum_{1 \le i < j \le N} \mathcal{L}_{\mathrm{Miwa}}([u], \lambda_i, \lambda_j)\, \eta_i \wedge \eta_j \approx \sum_{1 \le i < j \le N} \mathcal{L}_{i,j}[u]\, \mathrm{d}t_i \wedge \mathrm{d}t_j$$

References

Main reference:

- V. Continuum limits of pluri-Lagrangian systems. arXiv:1706.06830

Further reading:

- Wiersma, Capel. Lattice equations, hierarchies and Hamiltonian structures. Physica A. 1987
- Nijhoff, Hone, Joshi. On a Schwarzian PDE associated with the KdV hierarchy. Physics Letters A. 2000.
- Adler, Bobenko, Suris. Classification of integrable equations on quad-graphs. The consistency approach. Commun. Math. Phys. 2003.
- Lobb, Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009.
- Hietarinta, Joshi, Nijhoff. Discrete Systems and Integrability. (Chapter 12) Cambridge Texts in Applied Mathematics. 2016.
- Suris, V. On the Lagrangian structure of integrable hierarchies. In AI Bobenko (ed): Advances in Discrete Differential Geometry, Springer. 2016.

Relation to Hamiltonian formalism

Consider a pluri-Lagrangian two form $\sum_{i,j} \mathcal{L}_{ij} \mathrm{d}t_i \wedge \mathrm{d}t_j$ with

$$\mathcal{L}_{1j} = \frac{1}{2} v_x v_{t_j} - h_j(v_x, v_{xx}, \ldots)$$

and $\mathcal{L}_{ij}$ such that the multi-time Euler-Lagrange equations are

$$v_{t_j} = g_j(v_x, v_{xx}, \ldots) \qquad \text{with } g_j = \frac{\delta_1 h_j}{\delta v_x}$$

Introducing the variable $u = v_x$ we can write this as

$$u_{t_j} = \mathrm{D}_x g_j(u, u_x, \ldots).$$

This equation is Hamiltonian with Hamilton function h_j w.r.t. the Poisson bracket

$$\left\{ \textstyle\int f, \int g \right\} = \int \left(D_x \frac{\delta_1 f}{\delta u} \right) \frac{\delta_1 g}{\delta u}$$

on equivalence classes $(\int \cdot)$ mod x-derivatives.

If the pluri-Lagrangian two form $\sum_{i,j} \mathcal{L}_{ij} \mathrm{d}t_i \wedge \mathrm{d}t_j$ is closed on solutions, then the Hamiltonians are in involution: $\left\{ \int h_i, \int h_j \right\} = 0$

Relation to variational symmetries

[Petrera, Suris. Variational symmetries and pluri-Lagrangian systems in classical mechanics. arXiv:1710.01526.]

Consider a mechanical Lagrangian $L(q, q_t)$.

We say that a (generalized) vector field $V(q, q_t)$ is a variational symmetry if there exists a function $F(q, q_t)$, called the flux, such that

$$\mathrm{D}_V L(q, q_t) - \mathrm{D}_t F(q, q_t) = 0.$$

Noether's Theorem

If $V(q, q_t)$ is a variational symmetry with flux $F(q, q_t)$, then

$$J(q, q_t) = \frac{\partial L(q, q_t)}{\partial q_t} \cdot V(q, q_t) - F(q, q_t)$$

is an integral of motion.

Relation to variational symmetries

If we have a variational symmetry V with flux F and Noether integral J, then there is a pluri-Lagrangian one-form

$$\mathcal{L} = L_1(q, q_{t_1}, q_{t_2})\,\mathrm{d}t_1 + L_2(q, q_{t_1}, q_{t_2})\,\mathrm{d}t_2$$

with

$$\begin{aligned} L_1(q, q_{t_1}, q_{t_2}) &= L(q, q_{t_1}) \\ L_2(q, q_{t_1}, q_{t_2}) &= \frac{\partial L(q, q_{t_1})}{\partial q_{t_1}} \cdot q_{t_2} - J(q, q_{t_1}) \\ &= \frac{\partial L(q, q_{t_1})}{\partial q_{t_1}} \cdot (q_{t_2} - V(q, q_{t_1})) + F(q, q_{t_1}) \end{aligned}$$

which produces the equations of motion

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L}{\partial q_{t_1}} = 0 \qquad \text{and} \qquad q_{t_2} = V(q, q_{t_1})$$

If we have k commuting variational symmetries, we can produce a pluri-Lagrangian system in $k+1$ dimensions.