Modified equations for variational integrators

Mats Vermeeren

Technische Universität Berlin

Discretization in Geometry and Dynamics SFB Transregio 109

Berlin Mathematical School

Oberwolfach Workshop on
Geometric Numerical Integration

March 24, 2016

Lagrangian mechanics

Continuous: $\mathcal{L}(x(t), \dot{x}(t)) \quad \to \quad \frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0.$

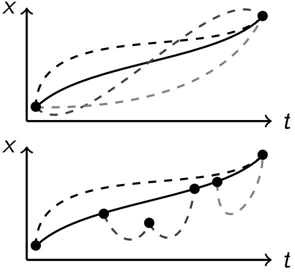
Discrete: $L_{\mathrm{disc}}(x_{j-1}, x_j) \quad \to \quad \mathrm{D}_2 L_{\mathrm{disc}}(x_{j-1}, x_j) + \mathrm{D}_1 L_{\mathrm{disc}}(x_j, x_{j+1}) = 0.$

Variational integrator: $L_{\mathrm{disc}}\big(x((t-h), x(t)) \quad \approx \quad \mathcal{L}(x(t), \dot{x}(t)).$

Are modified equations for variational integrators Lagrangian?

Using Taylor expansion and the Euler-MacLaurin formula we find $\mathcal{L}_{\mathrm{mesh}}[x(t)]$, a formal power series in h which satisfies

$$\int_a^b \mathcal{L}_{\mathrm{mesh}}[x(t)] = \sum_j L_{\mathrm{disc}}(x(jh), x(jh+h)).$$



meshed variational problem

Find critical curves of $\int_a^b \mathcal{L}[x(t)]\,\mathrm{d}t$ in the set of piecewise smooth curves that are consistent with a mesh of size h.

k-critical families of curves

A family of curves $x_h : [a, b] \to \mathbb{R}$ is k-critical if $\delta S_h = \mathcal{O}\big(h^{k+1} \, \|\delta x_h\|_1\big)$.

A family of admissible curves $x_h : [a, b] \to \mathbb{R}$ is k-critical iff

$$\frac{\delta \mathcal{L}_h}{\delta x} = \mathcal{O}(h^{k+1}) \qquad \text{and} \qquad \frac{\partial \mathcal{L}_h}{\partial x^{(\ell)}} = \mathcal{O}(h^{k+\ell+1}) \quad \forall \ell \geq 2.$$

Specifically for $\mathcal{L}_{\mathrm{mesh}}$:

$$\frac{\delta \mathcal{L}_{\mathrm{mesh}}}{\delta x} = \mathcal{O}(h^{k+1}) \qquad \Rightarrow \qquad \frac{\partial \mathcal{L}_{\mathrm{mesh}}}{\partial x^{(\ell)}} = \mathcal{O}(h^{k+\ell+1}) \quad \forall \ell \geq 2.$$

Modified Lagrangian

$$\mathcal{L}_{\mathrm{mod}}(x, \dot{x}) := \mathcal{L}_{\mathrm{mesh}}[x]\Big|_{\ddot{x} = f_h(x,\dot{x}),\ x^{(3)} = \frac{\mathrm{d}}{\mathrm{d}t} f_h(x,\dot{x}),\ \ldots}$$

where $\ddot{x} = f_h(x, \dot{x})$ is the modified equation.

Truncated mod eqn: $\ddot{x} = F_k^2(x, \dot{x}; h) + \mathcal{O}(h^{k+1}),\ x^{(3)} = F_k^3(x, \dot{x}; h) + \mathcal{O}(h^{k+1}), \ldots$

Truncated mod Lagrangian: $\mathcal{L}_{\mathrm{mod},k} = \mathcal{T}_k\left(\mathcal{L}_{\mathrm{mesh}}[x]\Big|_{x^{(j)} = F^j_{k-1}(x,\dot{x})} \right)$

The meshed modified Lagrangian $\mathcal{L}_{\mathrm{mesh}}[x]$ and the modified Lagrangian $\mathcal{L}_{\mathrm{mod},k}(x,\dot{x})$ have the same k-critical curves.

On $(k-1)$-critical curves:

$$\frac{\partial \mathcal{L}_{\mathrm{mod},k}}{\partial x} = \frac{\partial \mathcal{L}_{\mathrm{mesh}}}{\partial x} + \frac{\partial \mathcal{L}_{\mathrm{mesh}}}{\partial \ddot{x}} \frac{\partial F_k^2(x,\dot{x})}{\partial x} + \frac{\partial \mathcal{L}_{\mathrm{mesh}}}{\partial x^{(3)}} \frac{\partial F_k^3(x,\dot{x})}{\partial x} + \dots \bigg|_{x^{(j)} = F_{k-1}^j(x,\dot{x})}$$

$$= \frac{\partial \mathcal{L}_{\mathrm{mesh}}}{\partial x} + \mathcal{O}(h^{k+1}),$$

$$\frac{\partial \mathcal{L}_{\mathrm{mod},k}}{\partial \dot{x}} = \frac{\partial \mathcal{L}_{\mathrm{mesh}}}{\partial \dot{x}} + \mathcal{O}(h^{k+1}),$$

hence

$$\frac{\partial \mathcal{L}_{\mathrm{mod},k}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}_{\mathrm{mod},k}}{\partial \dot{x}} = \sum_{j=0}^{\infty} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j} \frac{\partial \mathcal{L}_{\mathrm{mesh}}}{\partial x^{(j)}} + \mathcal{O}(h^{k+1}).$$

Theorem

For a discrete Lagrangian L_{disc} that is a consistent discretization of some $\mathcal{L}$, the k-th truncation of the Euler-Lagrange equation of $\mathcal{L}_{\mathrm{mod},k}(x,\dot{x})$ is the k-th truncation of the modified equation.

Ref. M.V. Modified equations for variational integrators. `arXiv:1505.05411`