A Variational Structure for Integrable Hierarchies

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Integrable systems

An integrable system is a (system of) differential equation(s) with too much structure.

Lax Pairs,

Bi-Hamiltonian structure,

Commuting flows $\rightarrow$ hierarchy,

...

"An integrable system is a system that I can solve but you cannot."

Lagrangian PDEs

Lagrangian density $L(v, v_t, v_x, v_{tt}, v_{xt}, v_{xx}, \ldots)$

Action $\mathcal{S} = \int L \,\mathrm{d}x\,\mathrm{d}t$

Look for a function v that is a critical point of the action, i.e. for arbitrary infinitesimal variations δv:

$$0 = \delta\mathcal{S} = \int \delta L \,\mathrm{d}x\,\mathrm{d}t = \int \sum_I \frac{\partial L}{\partial v_I}\delta v_I \,\mathrm{d}x\,\mathrm{d}t$$

$$= \int \sum_I (-1)^{|I|}\,\mathrm{D}_I\left(\frac{\partial L}{\partial v_I}\right)\delta v \,\mathrm{d}x\,\mathrm{d}t$$

Euler-Lagrange equation:

$$\frac{\delta L}{\delta v} := \sum_I (-1)^{|I|}\,\mathrm{D}_I\left(\frac{\partial L}{\partial v_I}\right) = 0$$

Example of a Lagrangian PDE

Lagrangian density $L = \frac{1}{2} v_x v_t - v_x^3 - \frac{1}{2} v_x v_{xxx}$

Euler-Lagrange Equation:

$$
\begin{aligned}
0 = \frac{\delta L}{\delta v} &= \sum_I (-1)^{|I|} \mathrm{D}_I \left(\frac{\partial L}{\partial v_I} \right) \\
&= -\frac{1}{2} \mathrm{D}_t(v_x) - \frac{1}{2} \mathrm{D}_x(v_t) + 3 \mathrm{D}_x(v_x^2) + \frac{1}{2} \mathrm{D}_x(v_{xxx}) + \frac{1}{2} \mathrm{D}_{xxx}(v_x) \\
&= -v_{xt} + 6 v_x v_{xx} + v_{xxxx} \\
\Rightarrow v_{xt} &= 6 v_x v_{xx} + v_{xxxx}
\end{aligned}
$$

Substitute $u = v_x$ to find the Korteweg-de Vries equation

$$u_t = 6 u u_x + u_{xxx}.$$

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Pluri-Lagrangian systems

Multi-time $\mathbb{R}^N$, coordinates $(t_1, \ldots, t_N) = (x_1, \ldots x_{d-1}, t_d, \ldots, t_N)$

Field $u : \mathbb{R}^N \to \mathbb{R}$

Lagrangian d-form $\mathcal{L}(u, u_{t_1}, \ldots, u_{t_n}, u_{t_1 t_1}, u_{t_1 t_2}, \ldots, u_{t_n t_n})$

Definition

A field u solves the pluri-Lagrangian problem for $\mathcal{L}$ if

- *u is a critical point of the action $\int_S \mathcal{L}$ for all d-dimensional surfaces S in $\mathbb{R}^N$ simultaneously.*

The differential equations describing this condition are called the multi-time Euler-Lagrange equations.

Stepped Surfaces

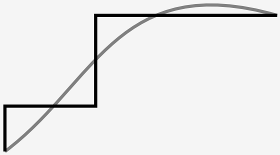
Definition

A d-dimensional coordinate surface is a surface S such that for distinct $i_1, \ldots, i_d$ and for all $x \in S$ we have $T_x S = \operatorname{span}\left(\frac{\partial}{\partial t_{i_1}}, \ldots, \frac{\partial}{\partial t_{i_d}}\right)$.

A stepped surface is a finite union of coordinate surfaces.

Lemma

If the action is stationary on every stepped surface, then it is stationary on every smooth surface.



Multi-time Euler-Lagrange equations for curves

Theorem

The multi-time Euler-Lagrange equations for the Lagrangian one-form $\mathcal{L} = \sum_i L_i \,\mathrm{d}t_i$ *are*

$$\frac{\delta_i L_i}{\delta u_I} = 0 \qquad \forall I \not\ni t_i,$$

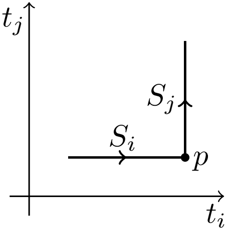
$$\frac{\delta_i L_i}{\delta u_{It_i}} = \frac{\delta_j L_j}{\delta u_{It_j}} \qquad \forall I,$$

where i *and* j *are distinct, and*

$$\frac{\delta_i L_i}{\delta u_I} := \sum_{\alpha \in \mathbb{N}} (-1)^\alpha \,\mathrm{D}_i^\alpha \frac{\partial L_i}{\partial u_{It_i^\alpha}} = \frac{\partial L_i}{\partial u_I} - \mathrm{D}_i \frac{\partial L_i}{\partial u_{It_i}} + \mathrm{D}_i^2 \frac{\partial L_i}{\partial u_{It_i^2}} - \ldots .$$

Multi-time Euler-Lagrange equations for curves

It is sufficient to look at a general L-shaped curve $S = S_i \cup S_j$.



The variation of the action on S_i is

$$\int_{S_i} (\delta L_i) \, \mathrm{d}t_i = \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta u_I} \delta u_I \, \mathrm{d}t_i + \sum_I \left(\frac{\delta_i L_i}{\delta u_{I t_i}} \right) \delta u_I \bigg|_p$$

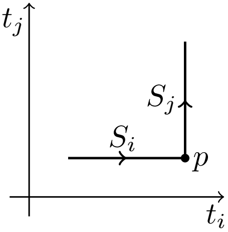
Multi-time Euler-Lagrange equations for curves

We have

$$\int_{S_i} (\delta L_i)\,\mathrm{d}t_i = \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta u_I} \delta u_I \,\mathrm{d}t_i + \sum_I \left(\frac{\delta_i L_i}{\delta u_{I t_i}} \delta u_I \right)\bigg|_p .$$

The other piece, S_j, contributes

$$\int_{S_j} (\delta L_j)\,\mathrm{d}t_j = \int_{S_j} \sum_{I \not\ni t_j} \frac{\delta_j L_j}{\delta u_I} \delta u_I \,\mathrm{d}t_j - \sum_I \left(\frac{\delta_j L_j}{\delta u_{I t_j}} \delta u_I \right)\bigg|_p .$$



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We have

$$\int_{S_i} (\delta L_i)\,\mathrm{d}t_i = \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta u_I} \delta u_I \,\mathrm{d}t_i + \sum_I \left(\frac{\delta_i L_i}{\delta u_{It_i}} \delta u_I \right) \bigg|_p .$$

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$$\int_{S_j} (\delta L_j)\,\mathrm{d}t_j = \int_{S_j} \sum_{I \not\ni t_j} \frac{\delta_j L_j}{\delta u_I} \delta u_I \,\mathrm{d}t_j - \sum_I \left(\frac{\delta_j L_j}{\delta u_{It_j}} \delta u_I \right) \bigg|_p .$$

Summing the two contributions we find

$$\int_S \delta \mathcal{L} = \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta u_I} \delta u_I \,\mathrm{d}t_i + \int_{S_j} \sum_{I \not\ni t_j} \frac{\delta_j L_j}{\delta u_I} \delta u_I \,\mathrm{d}t_j$$

$$+ \sum_I \left(\frac{\delta_i L_i}{\delta u_{It_i}} - \frac{\delta_j L_j}{\delta u_{It_j}} \right) \delta u_I \bigg|_p .$$

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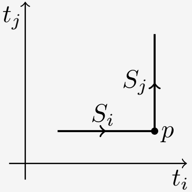


where i and j are distinct, and

$$\frac{\delta_i L_i}{\delta u_I} := \sum_{\alpha \in \mathbb{N}} (-1)^\alpha \, \mathrm{D}_i^\alpha \frac{\partial L_i}{\partial u_{It_i^\alpha}} = \frac{\partial L_i}{\partial u_I} - \mathrm{D}_i \frac{\partial L_i}{\partial u_{It_i}} + \mathrm{D}_i^2 \frac{\partial L_i}{\partial u_{It_i^2}} - \dots .$$

Multi-time Euler-Lagrange equations for 2d surfaces

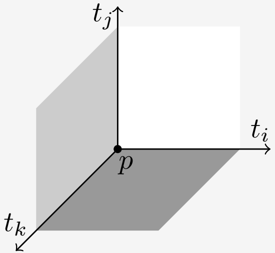
Theorem

The multi-time EL equations for $\mathcal{L} = \sum_{i<j} L_{ij} \,\mathrm{d}t_i \wedge \mathrm{d}t_j$ *are*

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I,$$



where i, j, *and* k *are distinct, and*

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} := \sum_{\alpha,\beta \in \mathbb{N}} (-1)^{\alpha+\beta} \,\mathrm{D}_i^\alpha \,\mathrm{D}_j^\beta \,\frac{\partial L_{ij}}{\partial u_{It_i^\alpha t_j^\beta}}.$$

The Korteweg-de Vries hierarchy

KdV hierarchy:

$$u_{t_k} = D_x(r_k[u])$$

$$\begin{aligned}
u_{t_1} &= u_x \\
u_{t_2} &= u_{xxx} + 6uu_x \qquad = \mathrm{D}_x(u_{xx} + 3u^2) \\
u_{t_3} &= u_{x^5} + 20u_x u_{xx} + 10uu_{xxx} + 30u^2 u_x \\
&= \mathrm{D}_x(u_{x^4} + 10u_x^2 + 10uu_{xx} + 10u^3) \\
&\vdots
\end{aligned}$$

Motivated by the first equation, we identify space with the first time-coordinate: $x \equiv t_1$.

The potential Korteweg-de Vries hierarchy

Potential v such that $v_x = u$, $\qquad g[v] := r[v_x]$.

KdV equations become:

$$v_{xt_k} = D_x(g_k[v])$$

Lagrangian: $L_k := \frac{1}{2} v_x v_t - h_k$, with $h_k = \frac{1}{4k+2} g_{k+1}$.

$$\frac{\delta L_k}{\delta v} = -v_{xt_k} + D_x(g_k[v])$$

PKdV hierarchy:

$$v_{t_k} = g_k[v]$$

Pluri-Lagrangian structure for PKdV hierarchy

Lagrangian two-form

$$\mathcal{L} = \sum_{i<j} L_{ij} \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

with coefficients:

$$L_{1i} = \frac{1}{2} v_x v_{t_i} - h_i$$

(the clasical Lagrangians)

$$L_{ij} = \frac{1}{2}(v_{t_i} g_j - v_{t_j} g_i) + (a_{ij} - a_{ji}) - \tfrac{1}{2}(b_{ij} - b_{ji})$$

(Obtained from the fact that the flows are variational symmetries of each other)

Multi-time Euler-Lagrange equations

- By construction, the equations $\frac{\delta_{1i} L_{1i}}{\delta v} = 0$ are

$$v_{xt_i} = \mathrm{D}_x\, g_i .$$

- The equation $\frac{\delta_{1i} L_{1i}}{\delta v_x} = \frac{\delta_{ij} L_{ji}}{\delta v_{t_j}}$ yields

$$v_{t_i} = g_i .$$

- All other Euler-Lagrange equations are corollaries of these.

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The multi-time EL equations for the Lagrangian two-form $\mathcal{L} = \sum_{i<j} L_{ij} \, \mathrm{d}t_i \wedge \mathrm{d}t_j$ with

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are the PKdV equations

$$v_{t_i} = g_i,$$

and corollaries thereof.

Multi-time Euler-Lagrange equations

The multi-time EL equations are the PKdV equations,

$$v_{t_i} = g_i,$$

and corollaries thereof.

Surprisingly, our variational method produces first order evolution equations. Discrepancy with:

- Classical Lagrangian formalism
- Discrete pluri-Lagrangian systems on quad graphs

References

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