A variational principle for discrete and continuous integrable systems

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Discretization in
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Mathematical
School

Contents

Pluri-Lagrangian formalism

(a.k.a. Lagrangian multiforms)

A variational description for many kinds of integrable systems:

	continuous	discrete
$d = 1$	ODEs	maps
$d \geq 2$	PDEs	PΔEs

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Variational analogue of $\{H_i, H_j\} = 0$

Integrable systems come in families:

finite (classical mechanics, ...) or infinite (Toda lattice, KdV equation,...) hierarchies of commuting equations.

On the Hamiltonian side, integrability is characterized by $\{H_i, H_j\} = 0$.

Consequence: ODEs for H_i and H_j commute: $\frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\mathrm{d}}{\mathrm{d}t_j} = \frac{\mathrm{d}}{\mathrm{d}t_j}\frac{\mathrm{d}}{\mathrm{d}t_i}$

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What about the Lagrangian side?

Pluri-Lagrangian principle ($d = 1$)

Combine the Lagrange functions $\mathcal{L}_i[u]$ into a Lagrangian 1-form

$$\mathcal{L}[u] = \sum_i \mathcal{L}_i[u] \, \mathrm{d}t_i .$$

Look for fields $u : \mathbb{R}^N \to \mathbb{C}$ that minimize the action

$$S_\Gamma = \int_\Gamma \mathcal{L}$$

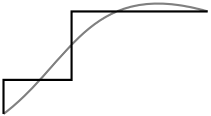
w.r.t. variations of u, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$

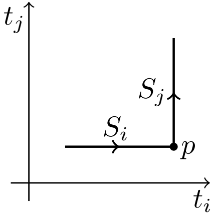
Multi-time Euler-Lagrange equations

Consider a Lagrangian one-form $\mathcal{L} = \sum_i \mathcal{L}_i[u] \, \mathrm{d}t_i$

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.





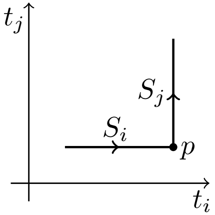
Variations are local, so it is sufficient to look at a general L-shaped curve $S = S_i \cup S_j$.

Multi-time Euler-Lagrange equations

The variation of the action on S_i is

$$\delta \int_{S_i} \mathcal{L}_i \,\mathrm{d}t_i = \int_{S_i} \sum_I \frac{\partial \mathcal{L}_i}{\partial u_I} \delta u_I \,\mathrm{d}t_i$$

$$= \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i \mathcal{L}_i}{\delta u_I} \delta u_I \,\mathrm{d}t_i + \sum_I \frac{\delta_i \mathcal{L}_i}{\delta u_{It_i}} \delta u_I \Big|_p ,$$



where I denotes a multi-index, and

$$\frac{\delta_i \mathcal{L}_i}{\delta u_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial \mathcal{L}_i}{\partial u_{It_i^\alpha}} = \frac{\partial \mathcal{L}_i}{\partial u_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial \mathcal{L}_i}{\partial u_{It_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial \mathcal{L}_i}{\partial u_{It_i^2}} - \ldots$$

Multi-time Euler-Lagrange equations for curves, $\mathcal{L} = \sum_i \mathcal{L}_i[u] \,\mathrm{d}t_i$

$$\frac{\delta_i \mathcal{L}_i}{\delta u_I} = 0 \qquad \forall I \not\ni t_i \qquad \text{and} \qquad \frac{\delta_i \mathcal{L}_i}{\delta u_{It_i}} = \frac{\delta_j \mathcal{L}_j}{\delta u_{It_j}} \qquad \forall I,$$

Example: Kepler Problem

The classical Lagrangian of a particle in the gravitational potential

$$\mathcal{L}_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

$$\mathcal{L}_2[q] = q_{t_1} \cdot q_{t_2} + (q_1 \times q) \cdot e,$$

into a pluri-Lagrangian 1-form $\mathcal{L}_1 \mathrm{d}t_1 + \mathcal{L}_2 \mathrm{d}t_2$ and consider $q = q(t_1, t_2)$.

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Multi-time Euler-Lagrange equations:

$$\frac{\delta_1 \mathcal{L}_1}{\delta q} = 0 \quad \Rightarrow \quad q_{t_1 t_1} = -\frac{q}{|q|^3} \qquad \text{(Keplerian motion)}$$

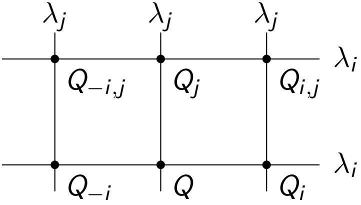
$$\frac{\delta_2 \mathcal{L}_2}{\delta q} = 0 \quad \Rightarrow \quad q_{t_1 t_2} = e \times q_1$$

$$\frac{\delta_2 \mathcal{L}_2}{\delta q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_2} = e \times q \qquad \text{(Rotation)}$$

$$\frac{\delta_1 \mathcal{L}_1}{\delta q_{t_1}} = \frac{\delta_2 \mathcal{L}_2}{\delta q_{t_2}} \quad \Rightarrow \quad q_{t_1} = q_{t_1}$$

Discrete pluri-Lagrangian principle ($d = 1$)

$Q : \mathbb{Z}^N \to M$ and L a discrete 1-form: $L\big(Q_i, Q, \lambda_i\big) = -L\big(Q, Q_i, \lambda_i\big)$



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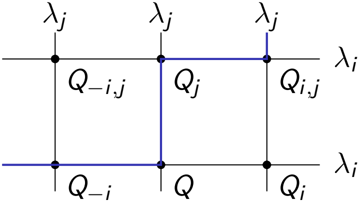
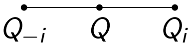


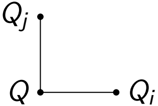
Discrete pluri-Lagrangian principle

Action sum is critical along any discrete curve in the lattice.

Discrete multi-time Euler-Lagrange equations

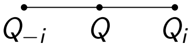


$$\frac{\partial}{\partial Q}\left(L(Q_{-i}, Q, \lambda_i) + L(Q, Q_i, \lambda_i)\right) = 0$$

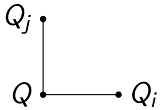


$$\frac{\partial}{\partial Q}\left(L(Q_j, Q, \lambda_j) + L(Q, Q_i, \lambda_i)\right) = 0$$

Discrete pluri-Lagrangian 1-forms



$$\frac{\partial}{\partial Q}\big(L(Q_{-i}, Q, \lambda_i) + L(Q, Q_i, \lambda_i)\big) = 0$$



$$\frac{\partial}{\partial Q}\big(L(Q_j, Q, \lambda_j) + L(Q, Q_i, \lambda_i)\big) = 0$$

If $Q : \mathbb{Z}^N \to M$ is a solution, then we can find P such that

$$P = -\frac{\partial}{\partial Q} L(Q_i, Q, \lambda_i) = \frac{\partial}{\partial Q} L(Q_{-i}, Q, \lambda_i) \qquad \text{for } i = 1, \ldots, N$$

Then

$$(Q, P) \mapsto (Q_i, P_i) \qquad \text{for } i = 1, \ldots, N$$

are commuting symplectic maps on T^*M.

Examples

- Discrete-time Toda lattice. $M = \mathbb{R}^n$
- Billiards in confocal quadrics. $M = S^n$ (unit velocities)
- ...

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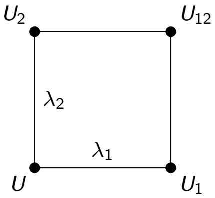
Quad equations

$$\mathcal{Q}(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2) = 0$$

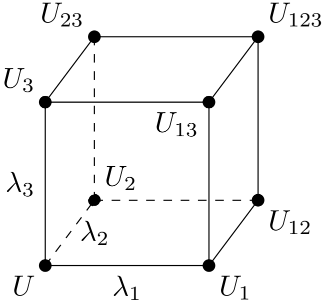
Subscripts of U denote lattice shifts, λ_1, λ_2 are parameters.
Invariant under symmetries of the square, affine in each of U, U_1, U_2, U_{12}.



Integrability for systems quad equations:
Multi-dimensional consistency of

$$\mathcal{Q}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) = 0,$$

i.e. the thrunderee ways of calculating U_{123} give the same result.



Classification (under some extra assumptions) by Adler, Bobenko and Suris (ABS List).

Example: lattice potential KdV:
$(U - U_{12})(U_1 - U_2) - \lambda_1 + \lambda_2 = 0$

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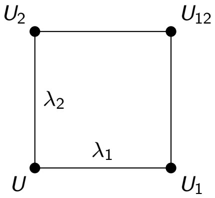

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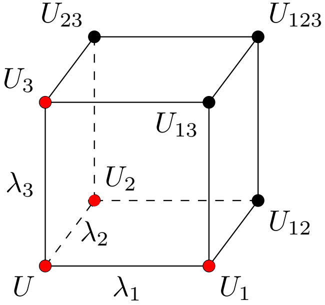

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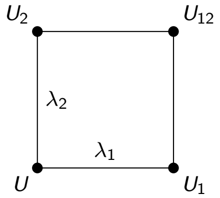

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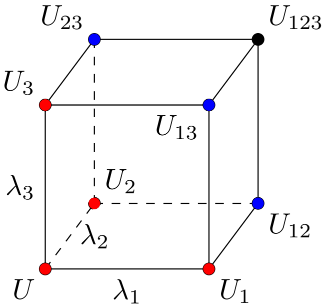

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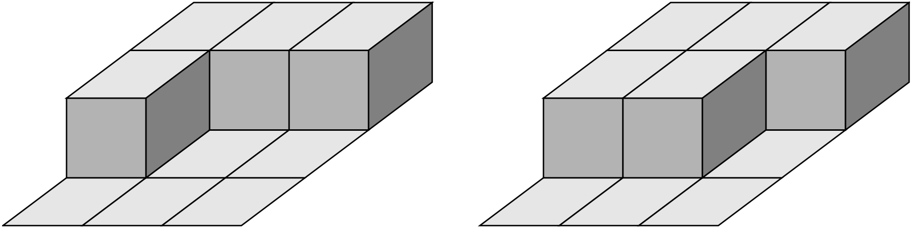
Example: lattice potential KdV:
$(U - U_{12})(U_1 - U_2) - \lambda_1 + \lambda_2 = 0$

Pluri-Lagrangian structure for quad equations

For some discrete 2-form

$$\mathcal{L}(\square_{ij}) = \mathcal{L}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j),$$

the action $\sum_{\square \in \Gamma} \mathcal{L}(\square)$ is critical on all 2-surfaces Γ in $\mathbb{N}^N$ simultaneously.



To derive Euler-Lagrange equations: vary U at each point individually.

$\hookrightarrow$ It is sufficient to consider corners of an elementary cube.

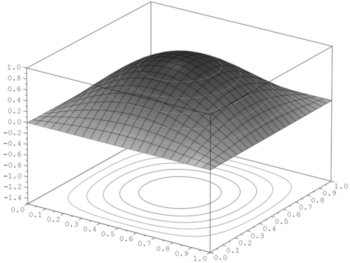
[Lobb, Nijhoff. 2009]

Pluri-Lagrangian principle ($d = 2$, continuous)

Given a 2-form

$$\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $u : \mathbb{R}^N \to \mathbb{C}$, such that $\int_\Gamma \mathcal{L}$ is critical on all smooth 2-surfaces Γ in multi-time $\mathbb{R}^N$.

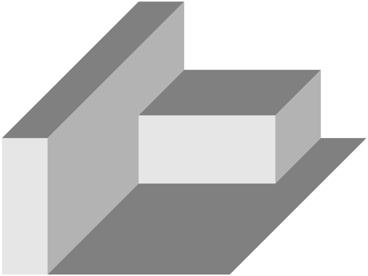


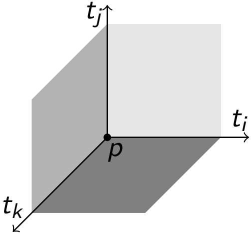
Example: KdV hierarchy, where $t_1 = x$ is the shared space coordinate, t_i time for i-th flow. (Details will follow.)

Multi-time EL equations

Consider a Lagrangian 2-form $\mathcal{L} = \sum_{i,j} \mathcal{L}_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$.

Every smooth surface can be approximated arbitrarily well by stepped surfaces. Hence it is sufficient to require criticality on stepped surfaces, or just on their elementary corners.





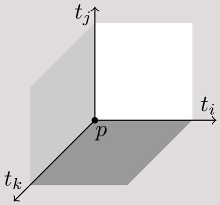
Multi-time EL equations

for $\mathcal{L} = \sum_{i,j} \mathcal{L}_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij}\mathcal{L}_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij}\mathcal{L}_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik}\mathcal{L}_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij}\mathcal{L}_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk}\mathcal{L}_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki}\mathcal{L}_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij}\mathcal{L}_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial \mathcal{L}_{ij}}{\partial u_{It_i^\alpha t_j^\beta}}$$

[Suris, V. 2016.]

Example: Potential KdV hierarchy

$$u_{t_2} = g_2[u] = u_{xxx} + 3u_x^2,$$
$$u_{t_3} = g_3[u] = u_{xxxxx} + 10u_x u_{xxx} + 5u_{xx}^2 + 10u_x^3,$$

where we identify $t_1 = x$.

The differentiated equations $u_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x} g_i[u]$ are Lagrangian with

$$\mathcal{L}_{12} = \frac{1}{2}u_x u_{t_2} - \frac{1}{2}u_x u_{xxx} - u_x^3,$$
$$\mathcal{L}_{13} = \frac{1}{2}u_x u_{t_3} - u_x u_{xxxxx} - 2u_{xx}u_{xxxx} - \frac{3}{2}u_{xxx}^2 + 5u_x^2 u_{xxx} + 5u_x u_{xx}^2 + \frac{5}{2}u_x^4.$$

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We choose the coefficient $\mathcal{L}_{23}$ of

$$\mathcal{L} = \mathcal{L}_{12}[u] \,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + \mathcal{L}_{13}[u] \,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + \mathcal{L}_{23}[u] \,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

such that the pluri-Lagrangian 2-form is closed on solutions (nontrivial task!). It is of the form

$$\mathcal{L}_{23} = \frac{1}{2}\left(u_{t_2} g_3[u] - u_{t_3} g_2[u]\right) + p_{23}[u].$$

Example: Potential KdV hierarchy

- The equations $\dfrac{\delta_{12}\mathcal{L}_{12}}{\delta u} = 0$ and $\dfrac{\delta_{13}\mathcal{L}_{13}}{\delta u} = 0$ yield

$$u_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x} g_2[u] \qquad \text{and} \qquad u_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x} g_3[u].$$

- The equations $\dfrac{\delta_{12}\mathcal{L}_{12}}{\delta u_x} = \dfrac{\delta_{32}\mathcal{L}_{32}}{\delta u_{t_3}}$ and $\dfrac{\delta_{13}\mathcal{L}_{13}}{\delta u_x} = \dfrac{\delta_{23}\mathcal{L}_{23}}{\delta u_{t_2}}$ yield

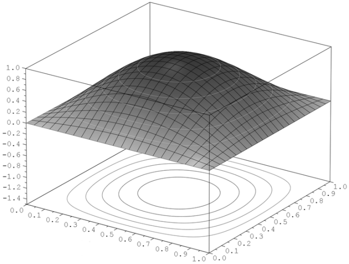
$$u_{t_2} = g_2 \qquad \text{and} \qquad u_{t_3} = g_3,$$

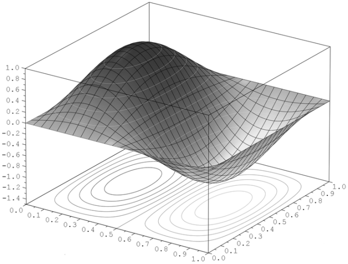
  the evolutionary equations!
- All other multi-time EL equations are corollaries of these.

Closedness of the Lagrangian form

One could require additionaly that $\mathcal{L}$ is closed on solutions
$\hookrightarrow$ "Lagrangian multiform systems".

Then the action is not just critical on every curve/surface, but also takes the same value on every curve/surface.





We do not take this as part of the definition, because one can show

Proposition

$\mathrm{d}\mathcal{L}$ is constant on the set of solutions.

Closedness relates to other notions of integrability

If $\mathrm{d}\left(\sum_i \mathcal{L}_i \, \mathrm{d}t_i \right) = 0$, then $\dfrac{\mathrm{d}\mathcal{L}_k}{\mathrm{d}t_j} = \dfrac{\mathrm{d}\mathcal{L}_j}{\mathrm{d}t_k}$

Variational symmetries

t_j-flow deforms $\mathcal{L}_k$ by a t_k-derivative.
$\Rightarrow$ Individual flows are variational symmetries of each other.

Same in higher dimensions.

Variational symmetries can be used to construct pluri-Lagrangian structures

- $d = 1$: [Petrera, Suris, 2017],
- $d = 2$: [Petrera, V, in preparation]

Hamiltonians in involution

- $d = 1$: Legendre transform and clever use of variational principle gives $\frac{\mathrm{d}H_k}{\mathrm{d}t_j} = \{H_j, H_k\} = 0$ [Suris, 2013]
- $d = 2$: [Suris, V, 2016] and work in progress.

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Miwa shifts

Continuum limit of an integrable difference equation

Skew embedding of the mesh $\mathbb{Z}^N$ into multi-time $\mathbb{R}^N$
Discrete Q is a sampling of the continuous q:

$$Q = Q(\mathbf{n}) = q(t_1, t_2, \dots, t_N),$$

$$Q_i = Q(\mathbf{n} + \mathfrak{e}_i) = q\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \dots, t_N + 2(-1)^N \frac{\lambda_i^N}{N} \right)$$

[Miwa. On Hirota's difference equations. Proceedings of the Japan Academy A. 1982]

Write quad equation in terms of q and expand in λ_1.

In the leading order, we only see t_1-derivatives of q, but we want to obtain PDEs.

$\hookrightarrow$ leading order cancellation required to get a meaningful result.

$\hookrightarrow$ whole hierarchy from single difference equation.

Continuum limit of the Lagrangian

- Using Miwa correspondence:

$$\text{Discrete } L \quad \rightarrow \quad \text{Power series } \mathcal{L}_{\mathrm{disc}}[u(\mathbf{t})]$$

Action for $\mathcal{L}_{\mathrm{disc}}[u(\mathbf{t})]$ is still a sum.

- Two applications of the Euler-Maclaurin formula:

$$\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=0}^{\infty} \frac{B_i B_j}{i! j!} \partial_1^i \partial_2^j \mathcal{L}_{\mathrm{disc}}([u], \lambda_1, \lambda_2).$$

where the differential operators are $\quad \partial_k = \sum_{j=1}^{N} (-1)^{j+1} \frac{2\lambda_k^j}{j} \frac{\mathrm{d}}{\mathrm{d}t_j}$

- Then there holds $L_{\mathrm{disc}}(\square) = \int_{\blacksquare} \mathcal{L}_{\mathrm{Miwa}}([u(\mathbf{t})], \lambda_1, \lambda_2)\, \eta_1 \wedge \eta_2,$

where η_1 and η_2 are the 1-forms dual to the Miwa shifts.

This suggests the Lagrangian 2-form

$$\sum_{1 \leq i < j \leq N} \mathcal{L}_{\mathrm{Miwa}}([u], \lambda_i, \lambda_j)\, \eta_i \wedge \eta_j.$$

Continuum limit of a Lagrangian 2-form

$L(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2)$ Suitable choice $\Rightarrow$ leading order cancellation

$\big\downarrow$ Miwa shifts, Taylor expansion

$\mathcal{L}_{\mathrm{disc}}([u], \lambda_1, \lambda_2)$

$\big\downarrow$ Euler-Maclaurin formula

$$\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=1}^{\infty} (-1)^{i+j} 4 \frac{\lambda_1^i}{i} \frac{\lambda_2^j}{j} \mathcal{L}_{i,j}[u]$$

$\big\downarrow$ $\qquad\qquad\qquad\qquad$ $\big\downarrow$

$$\sum_{1 \leq i < j \leq N} \mathcal{L}_{\mathrm{Miwa}}([u], \lambda_i, \lambda_j)\, \eta_i \wedge \eta_j = \sum_{1 \leq i < j \leq N} \mathcal{L}_{i,j}[u]\, \mathrm{d}t_i \wedge \mathrm{d}t_j$$

Continuum limits of ABS equations

$Q1_{\delta=0}$	$\rightarrow$	$v_3 = v_{111} - \dfrac{3v_{11}^2}{2v_1}$	Schwarzian KdV
$Q1_{\delta=1}$	$\rightarrow$	$v_3 = v_{111} - \dfrac{3}{2}\dfrac{v_{11}^2 - \frac{1}{4}}{v_1}$	
Q2	$\rightarrow$	$v_3 = v_{111} - \dfrac{3}{2}\dfrac{v_{11}^2 - \frac{1}{4}}{v_1} - \dfrac{3}{2}\dfrac{v_1^3}{v^2}$	
$Q3_{\delta=0}$	$\rightarrow$	$v_3 = v_{111} - \dfrac{3}{2}\dfrac{v_{11}^2 - \frac{1}{4}}{v_1} + \dfrac{1}{2}v_1^3$	
$Q3_{\delta=1}$	$\rightarrow$	$v_3 = v_{111} - \dfrac{3}{2}\dfrac{v_{11}^2 - \frac{1}{4}}{v_1} + \dfrac{1}{2}v_1^3 - \dfrac{3}{2}\dfrac{v_1^3}{\sin(v)^2}$	
Q4	$\rightarrow$	$v_3 = v_{111} - \dfrac{3}{2}\dfrac{v_{11} - \frac{1}{4}}{v_1} - \dfrac{3}{2}\wp(2v)v_1^3$	Krichever-Novikov
H1	$\rightarrow$	$v_3 = v_{111} + 3v_1^2$	Potential KdV
$H3_{\delta=0}$	$\rightarrow$	$v_3 = v_{111} + \dfrac{1}{2}v_1^3$	Potential mKdV

All with their hierarchies

Conclusions

- The pluri-Lagrangian (or Lagrangian multiform) principle is a widely applicable characterization of integrability:

  It applies to integrable ODEs and PDEs, and to integrable difference equations of any dimension.
- (Almost-)closedness of the pluri-Lagrangian form ($\mathrm{d}\mathcal{L} = \textit{const}$) links this pluri-Lagrangian system to the established theory of integrable systems.
- Discrete theory is better understood: continuum limits are a useful tool to develop the continuous theory.

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Thank you for your attention!