Continuum limits of pluri-Lagrangian systems

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Discretization in
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1. Discrete pluri-Lagrangian systems

2. Continuous pluri-Lagrangian systems

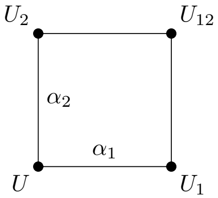
3. Continuum limits

Quad equations

Quad equation on $\mathbb{Z}^2$:

$$Q(U, U_1, U_2, U_{12}, \alpha_1, \alpha_2) = 0$$

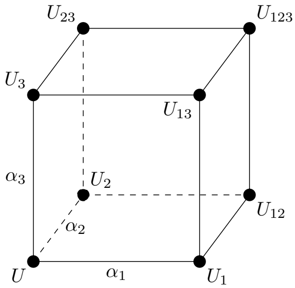
Subscripts of x denote lattice shifts, α_1, α_2 are parameters.
Invariant under symmetries of the square, affine in each of U, U_1, U_2, U_{12}.



Integrability for systems quad equations:
Multi-dimensional consistency of

$$Q(U, U_i, U_j, U_{ij}, \alpha_i, \alpha_j) = 0,$$

i.e. the three ways of calculating x_{123} give the same result.



Quad equations

- Classification multidimensionally consistent quad equations in the ABS list.

  [VE Adler, AI Bobenko, YB Suris. Classification of integrable equations on quad-graphs. The consistency approach. Commun. Math. Phys. 2003.]

- Variational formulation in which the Lagrangian is "an extended object capable of producing a multitude of consistent equations"
  $\hookrightarrow$ i.e. defined in the higher-dimensional lattice

  [S Lobb, F Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009.]

Discrete Pluri-Lagrangian problem

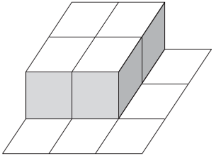
For some discrete 2-form

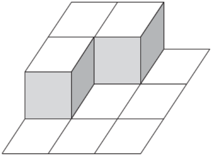
$$\mathcal{L}(\sigma_{ij}) = \mathcal{L}(U, U_i, U_j, U_{ij}, \alpha_i, \alpha_j),$$

find a field $U : \mathbb{Z}^N \to \mathbb{C}$ such that the action

$$\textstyle\sum_{\sigma_{ij} \in S} \mathcal{L}(\sigma_{ij})$$

is critical on all discrete 2-surfaces S in $\mathbb{Z}^N$ simultaneously.





Furthermore, the critical value of the action does not depend on the surface, i.e. the discrete 2-form $\mathcal{L}$ is closed on solutions.

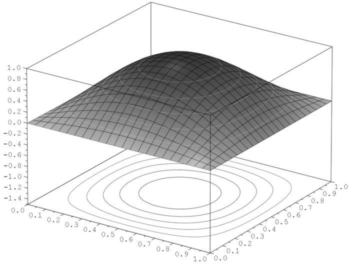
All ABS equations can be described this way.

Continuous Pluri-Lagrangian problem

Given a 2-form

$$\mathcal{L} = \sum_{i,j} L_{ij}[u]\,\mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $u : \mathbb{R}^N \to \mathbb{C}$, such that $\displaystyle\int_\Gamma \mathcal{L}$ is critical on all smooth 2-surfaces Γ in multi-time $\mathbb{R}^N$.

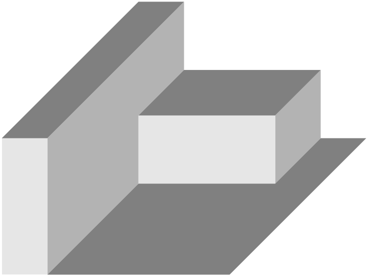


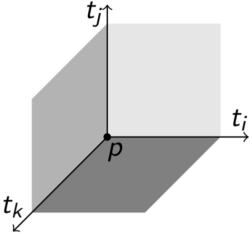
How to calculate Euler-Lagrange equations? Unlike the discrete case there are no elementary building blocks of smooth surfaces.

Multi-time EL equations

Consider a Lagrangian 2-form $\mathcal{L} = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$.

Every smooth surface can be approximated arbitrarily well by stepped surfaces. Hence it is sufficient to require criticality on stepped surfaces, or just on their elementary corners.





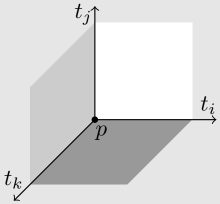
Multi-time EL equations

for $\mathcal{L} = \sum_{i,j} L_{ij}[u]\,\mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial u_{It_i^\alpha t_j^\beta}}$$

[YB Suris, MV. On the Lagrangian structure of integrable hierarchies. In AI Bobenko (ed): Advances in Discrete Differential Geometry, Springer. 2016.]

Continuum limit of H1 (lattice potential KdV)

$$\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + U_{1,2} - U\right)\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} + U_2 - U_1\right) = \frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2} \qquad \text{(lpKdV)}$$

This is a well-chosen representative of H1 out of many equivalent forms. Often one finds it written as $(X - X_{12})(X_2 - X_1) = \alpha_1 - \alpha_2$

Method by Wiersma and Capel produces the pKdV hierarchy from (lpKdV)

They used an differential-difference equation as intermediate step. Here we will present the same limit in a single step.

[GL Wiersma, HW Capel. Lattice equations, hierarchies and Hamiltonian structures. Physica A. 1987]

Continuum limit of H1 (lattice potential KdV)

Miwa variables

Skew embedding of the mesh $\mathbb{Z}^N$ into multi-time $\mathbb{R}^N$
Discrete $U : \mathbb{Z}^N \to \mathbb{C}$ is a sampling of the continuous $u : \mathbb{R}^N \to \mathbb{C}$:

$$U = U(\mathbf{n}) = u(t_1, t_2, \ldots, t_n),$$

$$U_i = U(\mathbf{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \ldots, t_n + 2(-1)^n \frac{\lambda_i^N}{N} \right)$$

[T Miwa. On Hirota's difference equations. Proceedings of the Japan Academy A. 1982]

Plug into $\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + U_{1,2} - U \right) \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} + U_2 - U_1 \right) = \frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2}$

and expand in λ_1, λ_2.

In leading order everything cancels (due to very specific form of quad eq).
$\hookrightarrow$ generically we would have an ODE in t_1.

Continuum limit of H1 (lattice potential KdV)

Series expansion

$$\text{Quad Eq.} \quad \to \quad \sum_{i,j} \frac{4}{ij} f_{i,j}[u]\, \lambda_1^i \lambda_2^j = 0,$$

where $f_{j,i} = -f_{i,j}$, the factor $\dfrac{4}{ij}$ is chosen to normalize the $f_{0,j}$, and

$$\begin{aligned}
f_{0,1} &= -u_{t_2}, \\
f_{0,2} &= -3u_{t_1}^2 - u_{t_1 t_1 t_1} - \frac{3}{2} u_{t_1 t_2} + u_{t_3}, \\
f_{0,3} &= 8u_{t_1} u_{t_1 t_1} + 4u_{t_1} u_{t_2} + \frac{4}{3} u_{t_1 t_1 t_1 t_1} - \frac{4}{3} u_{t_1 t_3} - u_{t_2 t_2} - u_{t_4}, \\
f_{0,4} &= -5u_{t_1 t_1}^2 - \frac{20}{3} u_{t_1} u_{t_1 t_1 t_1} + 10 u_{t_1} u_{t_1 t_2} + 5 u_{t_1 t_1} u_{t_2} - \frac{5}{4} u_{t_2}^2 - \frac{10}{3} u_{t_1} u_{t_3} \\
&\vdots
\end{aligned}$$

Continuum limit of H1 (lattice potential KdV)

Setting each f_{ij} equal to zero, we find

$$
\begin{aligned}
u_{t_2} &= 0, \\
u_{t_3} &= 3u_{t_1}^2 + u_{t_1 t_1 t_1} \\
u_{t_4} &= 0, \\
u_{t_5} &= 10u_{t_1}^3 + 5u_{t_1 t_1}^2 + 10u_{t_1} u_{t_1 t_1 t_1} + u_{t_1 t_1 t_1 t_1 t_1}, \\
&\vdots
\end{aligned}
$$

$\hookrightarrow$ pKdV hierarchy

Whole hierarchy from single quad equation

using Miwa correspondence

$$
\begin{aligned}
U &= U(\mathbf{n}) = u(t_1, t_2, \ldots, t_n), \\
U_i &= U(\mathbf{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \ldots, t_n + 2(-1)^n \frac{\lambda_i^N}{N} \right)
\end{aligned}
$$

Continuum limit of the Lagrangian for H1

A Lagrangian for (lpKdV) is

$$L(\square) = \frac{1}{2}\left(U - U_{i,j} - \lambda_i^{-1} - \lambda_j^{-1}\right)\left(U_i - U_j + \lambda_i^{-1} - \lambda_j^{-1}\right) \\ + \left(\lambda_i^{-2} - \lambda_j^{-2}\right)\log\left(1 + \frac{U_i - U_j}{\lambda_i^{-1} - \lambda_j^{-1}}\right).$$

Again this is a specific (and non-standard) choice among the many equivalent Lagrangians.

Using Miwa correspondence:

$$\text{Discrete } L \quad \to \quad \text{Power series } \mathcal{L}_{\mathrm{disc}}$$

Continuum limit of the Lagrangian

A series expansion is not the end of the story here. The action would still be a sum:

$$S = \sum_{\Box \in \sigma} L(\Box) \; = \; \sum_{\Box \in \sigma} \mathcal{L}_{\mathrm{disc}}[u(\text{point in } \Box)].$$

We want an integral

$$S = \int_\sigma \mathcal{L}.$$

Euler-MacLaurin formula

$$\begin{aligned}\sum_{j=0}^{n-1} g(j) &= \int_0^n g(t)\,\mathrm{d}t + \sum_{i=1}^{\infty} \frac{B_i}{i!}\Big(g^{(i-1)}(n) - g^{(i-1)}(0) \Big) \\ &= \int_0^n \left(g(t) + \sum_{i=1}^{\infty} \frac{B_i}{i!} g^{(i)}(t) \right) \mathrm{d}t,\end{aligned}$$

where the B_i are the Bernoulli numbers $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \cdots$.

Continuum limit of the Lagrangian

$$\mathcal{L}_{\mathrm{Miwa}}([u],\lambda_1,\lambda_2) = \sum_{i,j=0}^{\infty} \frac{B_i B_j}{i!j!} \partial_1^i \partial_2^j \mathcal{L}_{\mathrm{disc}}([u],\lambda_1,\lambda_2).$$

where the differential operators are $\partial_k = \sum_{j=1}^{N} (-1)^{j+1} \frac{\lambda_k^j}{j} \frac{\mathrm{d}}{\mathrm{d}t_j}$

$$L_{\mathrm{disc}}(\square) = \int_{M_{\lambda_1,\ldots,\lambda_N}(\blacksquare)} \mathcal{L}_{\mathrm{Miwa}}([u(\mathbf{t})],\lambda_i,\lambda_j)\,\mathrm{d}t_i \wedge \mathrm{d}t_j,$$

Theorem

Write $\mathcal{L}_{\mathrm{Miwa}}([u],\lambda_1,\lambda_2) = \sum_{i,j=1}^{\infty} \frac{\lambda_1^i}{i} \frac{\lambda_2^j}{j} \mathcal{L}_{ij}[u],$

then $\mathcal{L} = \sum_{1 \le i < j \le n} \mathcal{L}_{i,j}[u]\,\mathrm{d}t_i \wedge \mathrm{d}t_j$

is a pluri-Lagrangian structure.

Continuum limit of the Lagrangian

Write $\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=1}^{\infty} \frac{\lambda_1^i}{i} \frac{\lambda_2^j}{j} \mathcal{L}_{i,j}[u]$, then

$\mathcal{L} = \sum_{1 \leq i < j \leq n} \mathcal{L}_{i,j}[u] \, \mathrm{d}t_i \wedge \ldots \wedge \mathrm{d}t_j$ is a pluri-Lagrangian structure.

Proof. The d-form equals (up to a truncation error)

$$\mathcal{L} = \sum_{1 \leq i < j \leq N} \mathcal{L}_{\mathrm{Miwa}}([u], \lambda_i, \lambda_j) \, \eta_i \wedge \eta_j,$$

where the η_i are the 1-forms dual to $\mathfrak{v}_i = \left(\lambda_i, -\frac{\lambda_i^2}{2}, \ldots, \pm \frac{\lambda_i^N}{N} \right)$

$$\int_\Gamma \mathcal{L} = \sum_{\Gamma_{\mathrm{disc}}} L(\square) \qquad \text{if } \Gamma_{\mathrm{disc}} \mapsto \Gamma \text{ under the Miwa correspondence.}$$

These Γ form a sufficiently large class of surfaces to derive the multi-time EL equations. $\square$

Coefficients (after some post-limit simplifications)

$$\mathcal{L}_{1,2} = \tfrac{1}{2} u_1 u_2 \qquad \mathcal{L}_{1,3} = -u_1^3 + \tfrac{1}{2} u_{11}^2 + \tfrac{1}{2} u_1 u_3$$

$$\mathcal{L}_{1,4} = \tfrac{1}{2} u_1 u_4 \qquad \mathcal{L}_{1,5} = -\tfrac{5}{2} u_1^4 + 5 u_1 u_{11}^2 - \tfrac{1}{2} u_{111}^2 + \tfrac{1}{2} u_1 u_5$$

$$\mathcal{L}_{2,3} = -3u_1^2 u_2 + u_{11} u_{12} - u_{111} u_2 + \tfrac{1}{2} u_2 u_3$$

$$\mathcal{L}_{2,4} = \tfrac{1}{2} u_2 u_4$$

$$\begin{aligned}\mathcal{L}_{2,5} = {} & -10 u_1^3 u_2 + 10 u_1 u_{11} u_{12} - 5 u_{11}^2 u_2 - 10 u_1 u_{111} u_2 - u_{111} u_{112} + \\ & u_{1111} u_{12} - u_{11111} u_2 + \tfrac{1}{2} u_2 u_5\end{aligned}$$

$$\mathcal{L}_{3,4} = -u_{11} u_{14} + \tfrac{1}{2} u_3 u_4$$

$$\begin{aligned}\mathcal{L}_{3,5} = {} & 18 u_1^5 + 30 u_1^3 u_{111} - 10 u_1^3 u_3 + 6 u_{11}^2 u_{111} + 8 u_1 u_{111}^2 - 6 u_1 u_{11} u_{1111} + \\ & 3 u_1^2 u_{11111} + 10 u_1 u_{11} u_{13} - 5 u_{11}^2 u_3 - 10 u_1 u_{111} u_3 - \tfrac{1}{2} u_{1111}^2 + \\ & u_{111} u_{11111} - u_{111} u_{113} + u_{1111} u_{13} - u_{11} u_{15} - u_{11111} u_3 + \tfrac{1}{2} u_3 u_5\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{4,5} = {} & -10 u_1^3 u_4 + 10 u_1 u_{11} u_{14} - 5 u_{11}^2 u_4 - 10 u_1 u_{111} u_4 - u_{111} u_{114} + \\ & u_{1111} u_{14} - u_{11111} u_4 + \tfrac{1}{2} u_4 u_5\end{aligned}$$

Coefficients (after some post-limit simplifications)

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$$\mathcal{L}_{1,4} = \tfrac{1}{2} u_1 u_4 \qquad \mathcal{L}_{1,5} = -\tfrac{5}{2} u_1^4 + 5 u_1 u_{11}^2 - \tfrac{1}{2} u_{111}^2 + \tfrac{1}{2} u_1 u_5$$

$$\mathcal{L}_{2,3} = -3 u_1^2 u_2 + u_{11} u_{12} - u_{111} u_2 + \tfrac{1}{2} u_2 u_3$$

$$\mathcal{L}_{2,4} = \tfrac{1}{2} u_2 u_4$$

$$\mathcal{L}_{2,5} = -10 u_1^3 u_2 + 10 u_1 u_{11} u_{12} - 5 u_{11}^2 u_2 - 10 u_1 u_{111} u_2 - u_{111} u_{112} + u_{1111} u_{12} - u_{11111} u_2 + \tfrac{1}{2} u_2 u_5$$

$$\mathcal{L}_{3,4} = -u_{11} u_{14} + \tfrac{1}{2} u_3 u_4$$

$$\mathcal{L}_{3,5} = 18 u_1^5 + 30 u_1^3 u_{111} - 10 u_1^3 u_3 + 6 u_{11}^2 u_{111} + 8 u_1 u_{111}^2 - 6 u_1 u_{11} u_{1111} + 3 u_1^2 u_{11111} + 10 u_1 u_{11} u_{13} - 5 u_{11}^2 u_3 - 10 u_1 u_{111} u_3 - \tfrac{1}{2} u_{1111}^2 + u_{111} u_{11111} - u_{111} u_{113} + u_{1111} u_{13} - u_{11} u_{15} - u_{11111} u_3 + \tfrac{1}{2} u_3 u_5$$

$$\mathcal{L}_{4,5} = -10 u_1^3 u_4 + 10 u_1 u_{11} u_{14} - 5 u_{11}^2 u_4 - 10 u_1 u_{111} u_4 - u_{111} u_{114} + u_{1111} u_{14} - u_{11111} u_4 + \tfrac{1}{2} u_4 u_5$$

Other equations

Q1: $$\frac{U_2 - U}{\lambda_2}\frac{U_{1,2} - U_1}{\lambda_2} - \frac{U_1 - U}{\lambda_1}\frac{U_{1,2} - U_2}{\lambda_1} = 0$$

produces the Schwarzian KdV hierarchy

$$\begin{aligned}
&\frac{u_{t_2}}{u_{t_1}} = 0, \\
&\frac{u_{t_3}}{u_{t_1}} = -\frac{3u_{t_1 t_1}^2}{2u_{t_1}^2} + \frac{u_{t_1 t_1 t_1}}{u_{t_1}}, \\
&\frac{u_{t_4}}{u_{t_1}} = 0, \\
&\frac{u_5}{u_{t_1}} = -\frac{45u_{t_1 t_1}^4}{8u_{t_1}^4} + \frac{25u_{t_1 t_1}^2 u_{t_1 t_1 t_1}}{2u_{t_1}^3} - \frac{5u_{t_1 t_1 t_1}^2}{2u_{t_1}^2} - \frac{5u_{t_1 t_1} u_{t_1 t_1 t_1 t_1}}{u_{t_1}^2} + \frac{u_{t_1 t_1 t_1 t_1 t_1}}{u_{t_1}}, \\
&\qquad \vdots
\end{aligned}$$

including a pluri-Lagrangian structure

Other equations

- $H3_{\delta=0}$: produces the potential modified KdV hierarchy

  But so far, no Lagrangian has been found that allows a series expansion.

- No results for other ABS equations at this moment.

- 1-form case:

  Fully discrete Toda $\rightarrow$ Toda hierarchy works perfectly.

References

Main:

- MV. Continuum limits of pluri-Lagrangian systems. arXiv:1706.06830

Further reading:

- GL Wiersma, HW Capel. Lattice equations, hierarchies and Hamiltonian structures. Physica A. 1987
- VE Adler, AI Bobenko, YB Suris. Classification of integrable equations on quad-graphs. The consistency approach. Commun. Math. Phys. 2003.
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- R Boll, M Petrera, YB Suris. What is integrability of discrete variational systems? Proc. R. Soc. A. 2014.
- J Hietarinta, N Joshi, FW Nijhoff. Discrete Systems and Integrability. (Chapter 12) Cambridge Texts in Applied Mathematics. 2016.
- YB Suris, MV. On the Lagrangian structure of integrable hierarchies. In AI Bobenko (ed): Advances in Discrete Differential Geometry, Springer. 2016.

Continuum limit of the Lagrangian for H1

The first row of coefficients is (with $u_k = u_{t_k}$)

$$
\begin{aligned}
\mathcal{L}_{12} &= \frac{1}{2} u_1 u_2 \\
\mathcal{L}_{13} &= -u_1^3 - \frac{1}{2} u_1 u_{111} + \frac{3}{8} u_2^2 + \frac{1}{2} u_1 u_3 \\
\mathcal{L}_{14} &= -2 u_1^2 u_2 - \frac{2}{3} u_1 u_{112} - \frac{1}{3} u_{11} u_{12} - \frac{1}{3} u_{111} u_2 + \frac{2}{3} u_2 u_3 + \frac{1}{2} u_1 u_4 \\
\mathcal{L}_{15} &= \frac{5}{3} u_1 u_{11}^2 - \frac{5}{4} u_1 u_2^2 - \frac{5}{3} u_1^2 u_3 + \frac{5}{18} u_{11} u_{1111} + \frac{1}{18} u_1 u_{11111} - \frac{5}{9} u_1 u_{113} \\
&\quad - \frac{5}{12} u_{12}^2 - \frac{5}{24} u_1 u_{122} - \frac{5}{18} u_{11} u_{13} - \frac{5}{12} u_{112} u_2 - \frac{5}{24} u_{11} u_{22} \\
&\quad - \frac{5}{18} u_{111} u_3 + \frac{5}{18} u_3^2 + \frac{5}{8} u_2 u_4 + \frac{1}{2} u_1 u_5 \\
&\vdots
\end{aligned}
$$

Continuum limit of the Lagrangian for H1

The first row of coefficients is (with $u_k = u_{t_k}$)

$$\mathcal{L}_{12} = \frac{1}{2} u_1 u_2$$

$$\mathcal{L}_{13} = -u_1^3 - \frac{1}{2} u_1 u_{111} + \frac{3}{8} {u_2}^2 + \frac{1}{2} u_1 u_3$$

$$\mathcal{L}_{14} = -2 u_1^2 u_2 - \frac{2}{3} u_1 u_{112} - \frac{1}{3} u_{11} u_{12} - \frac{1}{3} u_{111} u_2 + \frac{2}{3} u_2 u_3 + \frac{1}{2} u_1 u_4$$

$$\begin{aligned}\mathcal{L}_{15} = {} & \frac{5}{3} u_1 u_{11}^2 - \frac{5}{4} u_1 {u_2}^2 - \frac{5}{3} u_1^2 u_3 + \frac{5}{18} u_{11} u_{1111} + \frac{1}{18} u_1 u_{11111} - \frac{5}{9} u_1 u_{113} \\ & - \frac{5}{12} {u_{12}}^2 - \frac{5}{24} u_1 u_{122} - \frac{5}{18} u_{11} u_{13} - \frac{5}{12} u_{112} u_2 - \frac{5}{24} u_{11} u_{22} \\ & - \frac{5}{18} u_{111} u_3 + \frac{5}{18} {u_3}^2 + \frac{5}{8} u_2 u_4 + \frac{1}{2} u_1 u_5\end{aligned}$$

$$\vdots$$

How to get rid of these alien derivatives?

Eliminating alien derivatives

A derivative u_{t_k} or u_{It_k} with $k \neq 1, i, j$ is called alien to $\mathcal{L}_{i,j}$

Naive substitution

Let $\overline{\mathcal{L}}_{i,j}$ be the expression obtained from $\mathcal{L}_{i,j}$ by eliminating all alien derivatives using the pKdV equations.
Let $\overline{\mathcal{L}}$ denote the corresponding 2-form.

In general, substitutions in the Lagrangian using the EL equations can affect the EL equations!

Theorem

The Lagrangian obtained by naive substitution is equivalent to the original Lagrangian

Proof idea. By explicit calculation using the multi-time Euler-Lagrange equations for $\mathcal{L}$, we can write $\delta \overline{\mathcal{L}} = \mathrm{d}(\cdots)$. $\square$

Other equations

H3$_{\delta=0}$: produces the potential modified KdV hierarchy:

$$\frac{UU_1 - U_2 U_{12}}{\lambda_2} - \frac{UU_2 - U_1 U_{12}}{\lambda_1} = 0 \quad \rightarrow \quad u_{t_3} = u_{t_1 t_1 t_1} - 3\frac{u_{t_1} u_{t_1 t_1}}{u}, \ \dots$$

$$U = e^V \quad \updownarrow \quad u = e^v$$

$$\frac{e^{V+V_1} - e^{V_2 - V_{12}}}{\lambda_2} - \frac{e^{V+V_2} - e^{V_1 - V_{12}}}{\lambda_1} = 0 \quad \rightarrow \quad v_{t_3} = v_{t_1 t_1 t_1} - 2v_{t_1}^3, \ \dots$$

but so far, no Lagrangian has been found that allows a series expansion.

No results for other ABS equations at this moment.

1-form case: fully discrete Toda $\rightarrow$ Toda hierarchy works perfectly.