Structure-preserving discretisation of time-reparametrized Hamiltonian systems with application to nonholonomic mechanics

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Conformal Hamiltonian System (= time-reparameterized Hamiltonian system):

$$\dot{q} = N(q,p)\frac{\partial H(q,p)}{\partial p}, \qquad \dot{p} = -N(q,p)\frac{\partial H(q,p)}{\partial q} \qquad \boxed{i_X \Omega = N\, dH}$$

with Hamiltonian H and conformal factor N

→ Some non-holonomic systems can be put in conformal Hamiltonian form. (Hamiltonizable G-Chaplygin systems).

Properties of CHS:

→ $\mathcal{L}_X H = 0$

→ $\mathcal{L}_X \Omega = dN \wedge dH$

→ Volume form $\mu = N^{-1}\Omega^n$ is invariant: $\mathcal{L}_X \mu = 0$

Goal: construct measure-preserving integrators...

Focus on the invariant level set $\{H(q,p)=E\}$.

Altered Hamiltonian: $K_E(q,p) = N(q,p)\,(H(q,p)-E)$.

X_{K_E} its canonical Hamiltonian vector field: $i_{X_{K_E}}\Omega = dK_E$

$\rightarrow$ On $\{H=E\} = \{K_E=0\}$: $X_{K_E} = X$

Indeed: $dK_E \overset{H=E}{=} N\,dH \implies i_{X_{K_E}}\Omega = N\,dH = i_X\Omega$

$\rightarrow$ Can apply a symplectic integrator to the altered system X_{K_E} :

$\Psi_{h,E}: (q_j, p_j) \mapsto (q_{j+1}, p_{j+1})$, where E is the initial energy $E = H(q_0,p_0) \iff K_E(q_0,p_0)=0$

$\rightarrow$ Used by [Fernandez, Bloch, Olver, 2012] for discretizing nonholonomic systems
by [Hairer, 1997], [Reich, 1999] for variable step-size integrators

New approach: E parameter to be fixed later.

$\Psi_{h,E}$ symplectic $\longrightarrow$ modified Hamiltonian (in the sense of BEA)

$$K_{mod}(q,p;h,E) = K_E(q,p) + h\,K_1(q,p;E) + h^2 K_2(q,p;E) + \dots$$

Truncation $K^{(\ell)}_{mod}(q,p;h,E) = K_E(q,p) + \dots + h^\ell K_\ell(q,p;E)$

Take E such that $K_{mod}=0$ instead of $K=0$

Use $E = \mathcal{E}(q,p;h)$ with

$$\mathcal{E}(q,p;h) = \mathcal{E}_0(q,p) + h\,\mathcal{E}_1(q,p) + h^2\,\mathcal{E}_2(q,p) + \dots$$

defined by $K_{mod}(q,p;h,\mathcal{E}(q,p;h)) = 0$ $\qquad (\Rightarrow \mathcal{E}_0 = H)$

Proposed discretization $\Phi_h^{(\ell)}(q,p) = \Psi_{h,\mathcal{E}^{(\ell)}(q,p,h)}(q,p)$

($\Phi_h^{(0)}$ coincides with $\Psi_{h,E}$)

Thm Up to a local error of $O(h^{\ell+2})$ the conformal Hamiltonian system

$$\dot{p} = \frac{\partial K_{mod}^{(\ell)}}{\partial E}(q,p;h,\mathcal{E}^{(\ell)}(q,p;h))\,\frac{\partial \mathcal{E}^{(\ell)}}{\partial q}(q,p;h)$$

$$\dot{q} = \underbrace{-\frac{\partial K_{mod}^{(\ell)}}{\partial E}(q,p;h,\mathcal{E}^{(\ell)}(q,p;h))}_{= N_{mod}^{(\ell)}(q,p;h)}\,\frac{\partial \mathcal{E}^{(\ell)}}{\partial p}(q,p;h)$$

$\mathcal{E}^{(\ell)}$: modified conformal Hamiltonian

interpolates the discrete flow of $\Phi_h^{(\ell)}$.

Corollary Up to a local error of $O(h^{\ell+2})$ the map $\Phi_h^{(\ell)}$ preserves the modified measure

$$\mu_{mod}^{(\ell)} = (N_{mod}^{(\ell)})^{-1}\,\Omega^n.$$