Master's Thesis

On the pluri-Lagrangian structure of the KdV hierarchy

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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und eigenhändig sowie ohne unerlaubte fremde Hilfe und ausschlielich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

Die selbständige und eigenständige Anfertigung versichert an Eides statt:

Berlin, den 31. Oktober 2014

Mats Vermeeren

Acknowledgments

It's better to stand by someone's side than by yourself.

Jack London

There are many people who have contributed to the creation this thesis. Whether that was directly or indirectly, knowingly or not, I would like to thank them all. I feel a few of them should be mentioned explicitly.

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Finally I would like to thank my family for encouraging me to follow my interests and for not complaining too much when I decided to move abroad. Special thanks goes out to my brother Stijn for introducing me to beautiful bits of mathematics around the time high school math classes were boring me to death, and thus sparking my interest in mathematical studies.

Contents

1. Introduction

This thesis explores the concept of pluri-Lagrangian systems and investigates if it is a reasonable characterization of integrability for a hierarchy of variational differential equations. This notion was suggested by Suris [16] and already investigated in detail for discrete variational systems by Boll, Petrera and Suris [5], building on work of Lobb and Nijhoff [10]. Before discussing pluri-Lagrangian systems, let us quickly review traditional variational methods and spend a few words on the notion of integrable systems.

1.1. Variational methods

Nature is thrifty in all its actions.

Pierre Louis Maupertuis[1]

The solutions of most differential equations that occur in physics have a remarkable property. They minimize an abstract quantity called the action. A necessary condition for this to happen is that for any infinitesimal variation of the solution the corresponding variation of the action is zero. This condition can be used to find solutions.

Variations

Consider a vector bundle $X = \mathbb{R}^N \times \mathbb{R}^M$ over $\mathbb{R}^N$ and its n-th jet bundle $J^n X$. Let $\mathcal{L} : J^n X \to \Omega^N(\mathbb{R}^N)$ be smooth a form-valued function. In other words, $\mathcal{L}$ is a N-form that depends on a function $u : \mathbb{R}^N \to \mathbb{R}^M$ and its partial derivatives up to order n. We sometimes emphasize this dependence by writing $\mathcal{L}[u]$. We call $u = (u_1, \ldots, u_M)$ the *field* and $\mathcal{L}$ the *Lagrangian N-form*. The *action* on an open submanifold $S \subset \mathbb{R}^N$ is defined by

$$\mathfrak{S} := \int_S \mathcal{L}[u] \mathrm{d}V.$$

One way to formalize the idea of infinitesimal variations would be to introduce the

[1] Common, though far from exact, translation of *"la Nature dans la production de ses effets agit toûjours par les voies les plus simples"*.

variation as a function $\delta\mathcal{L} : J^nX \times J^nX \to \mathbb{R}^M$ that satisfies

$$\forall \delta u \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^M) : \quad \mathcal{L}[u + \varepsilon\delta u] = \mathcal{L}[u] + \varepsilon\delta\mathcal{L}[u, \delta u] + o(\varepsilon).$$

This has the advantage of being immediately understandable, but the theory will turn out nicer if we use the so-called *variational bicomplex*, which is introduced in Appendix A.1.

In this context, *variations* $\mathscr{V}$ of the field u are "vertical" vector fields, and $\delta\mathcal{L}$ is the "vertical" exterior derivative of $\mathcal{L}$. The corresponding variation of the action is given by

$$\delta\mathfrak{S} = \int_S \iota_{\operatorname{pr}\mathscr{V}}\, \delta\mathcal{L},$$

where ι denotes the interior product and $\operatorname{pr}\mathscr{V}$ the n-th jet prolongation of $\mathscr{V}$. We want $\delta\mathfrak{S}$ to be zero for any open submanifold $S \subset \mathbb{R}^N$ and any $\mathscr{V}$ with compact support in S. If this is the case we say that the action is *critical* or *stationary*.

The Euler-Lagrange equations

Consider coordinates $(t_1, \ldots, t_N)$ on $\mathbb{R}^N$. We will use the multi-index notation for partial derivatives: for any multi-index $I = (i_1, \ldots, i_N)$ we set

$$(u_k)_I := \frac{\partial^{|I|} u_k}{(\partial t_1)^{i_1} \ldots (\partial t_N)^{i_N}},$$

where $|I| = i_1 + \ldots + i_N$. The notations Ik and Ik^α will represent the multi-indices $(i_1, \ldots, i_k + 1, \ldots i_N)$ and $(i_1, \ldots, i_k + \alpha, \ldots i_N)$ respectively. When convenient we will also use the notations It_k and It_k^α for these multi-indices. We will write $k \notin I$ (or $t_k \notin I$) if $i_k = 0$ and $k \in I$ (or $t_k \in I$) if $i_k > 0$. We will denote by D_i or D_{t_i} the total derivative with respect to coordinate direction t_i,

$$\mathrm{D}_i := \mathrm{D}_{t_i} := \sum_{k=1}^N \sum_I (u_k)_{Ii} \frac{\partial}{\partial (u_k)_I},$$

and by $\mathrm{D}_I := \mathrm{D}_{t_1}^{i_1} \ldots \mathrm{D}_{t_1}^{i_N}$ the corresponding higher order derivatives.

Write $\mathcal{L} = L\, \mathrm{d}V$, where $\mathrm{d}V$ is a volume form that does not depend on the field u or its derivatives. Then we have $\delta(\mathrm{d}V) = 0$, hence there holds

$$\delta\mathcal{L} = \delta L \wedge \mathrm{d}V + L\, \delta(\mathrm{d}V) = \delta L \wedge \mathrm{d}V.$$

If the coefficient L of $\mathcal{L} = L\,\mathrm{d}V$ is polynomial in u and its derivatives, we have

$$\delta\mathfrak{S} = \int_S \iota_{\operatorname{pr}\mathscr{V}}\,\delta\mathcal{L} = \int_S \sum_{k=1}^{N}\sum_I \frac{\partial L}{\partial(u_k)_I}\delta(u_k)_I(\mathscr{V})\,\mathrm{d}V,$$

where the second sum is taken over all multi-indices, including the empty one, and $\delta(u_k)_I(\mathscr{V}) := \iota_{\operatorname{pr}\mathscr{V}}\,\delta(u_k)_I$ is the vertical one-form $\delta(u_k)_I$ evaluated on the prolonged variation $\operatorname{pr}\mathscr{V}$. Integration by parts gives us

$$\delta\mathfrak{S} = \int_S \sum_{k=1}^{N}\sum_I (-1)^{|I|}\,\mathrm{D}_I\left(\frac{\partial L}{\partial(u_k)_I}\right)\delta u_k(\mathscr{V})\,\mathrm{d}V.$$

There is no boundary term because $\mathscr{V}$ vanishes on the boundary of S. We see that $\delta\mathfrak{S}$ is zero for any variation $\mathscr{V}$ if and only if

$$\frac{\delta L}{\delta u_k} := \sum_I (-1)^{|I|}\,\mathrm{D}_I\left(\frac{\partial L}{\partial(u_k)_I}\right) = 0 \qquad \text{for } k = 1,\dots,M. \tag{1.1}$$

These are the *Euler-Lagrange equations*. The quantity on the left hand side of Equation (1.1) is called the *variational derivative* of L with respect to u_k. The variational derivative is characterized by the property that $\int \delta L \wedge \mathrm{d}V = \int \sum_k \frac{\delta L}{\delta u_k}\delta u_k \wedge \mathrm{d}V$.

The theory outlined above is called *Lagrangian field theory*. The most famous part of it is *Lagrangian mechanics*. In this case we have $N = 1$, so X is a vector bundle over the real line $\mathbb{R}$, which represents time and has coordinate t. Furthermore, the Lagrangian depends only on the first jet bundle, so the Euler-Lagrange equations (1.1) reduce to

$$\frac{\partial L}{\partial u_k} - \mathrm{D}_t\,\frac{\partial L}{\partial(u_k)_t} = 0 \qquad \text{for } k = 1,\dots,M. \tag{1.2}$$

In the rest of this text, we will generally have $N > 1$. On the other hand, we will restrict to $M = 1$, so the field u will be a real-valued function.

Literature

An excellent introduction to Lagrangian mechanics, along with other topics in classical mechanics, can be found in the well-known book by Arnold [2]. For more about Lagrangian field theory and the variational bicomplex, see e.g. Dickey [6, Chapter 19].

1.2. Integrable systems

An integrable system is a system that I can solve but you cannot.

Unknown origin[2]

The field of integrable systems is vast and diverse. There is not even a general definition of the concept of integrability. Roughly speaking, an integrable system is a system of differential equations (or a single one) that possesses some non-obvious structure which allows one either to find an exact solution, or to derive properties of solutions. Without knowledge of this structure, the problem is usually very inaccessible.

Hamiltonian systems

In many cases, the hidden structure of an integrable system is uncovered using the language of *Hamiltonian mechanics*. A *canonical Hamiltonian system* consists of the differential equations

$$\dot{q_k} = \frac{\partial H}{\partial p_k}, \qquad \dot{p_k} = -\frac{\partial H}{\partial q_k}, \qquad \text{for } k = 1, \ldots, N \tag{1.3}$$

where $H : \mathbb{R}^{2N} \to \mathbb{R} : (q_1, \ldots, q_N, p_1, \ldots, p_N) \mapsto H(q_1, \ldots, q_N, p_1, \ldots, p_N)$ is the *Hamilton function* or *Hamiltonian*, and the dot represents the time derivative. These equations are closely related to the equations (1.2) of Lagrangian mechanics. The Hamiltonian can be obtained from the corresponding Lagrangian through the so-called *Legendre transform*.

The equations of motion (1.3) can be written in a coordinate-free way by introducing the *Poisson bracket*

$$\{F, G\} := \sum_{k=1}^{N} \left(\frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k} - \frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} \right). \tag{1.4}$$

Then the time evolution of any function $F : \mathbb{R}^{2N} \to \mathbb{R} : (q_1, \ldots, q_N, p_1, \ldots, p_N) \mapsto F(q_1, \ldots, q_N, p_1, \ldots, p_N)$ is given by

$$\dot{F} = \{H, F\}. \tag{1.5}$$

In particular, F is a conserved quantity if and only if $\{H, F\} = 0$. If this is the case we say that F and H are *in involution*. Note that this also implies that H is a conserved quantity of the Hamiltonian flow of F, since the Poisson bracket is anti-symmetric.

[2] I heard this quote from Prof. Dr. Thomas Kriecherbauer during a summer school at TU Munich in July 2014. He did not know its origin and I have not been able to find out.

It is in this framework that the best-known definition of integrability is stated. A flow with Hamiltonian $H : \mathbb{R}^{2N} \to \mathbb{R}$ is *completely integrable in the sense of Liouville-Arnold* if there exist functions $H_2, \ldots, H_N : \mathbb{R}^{2N} \to \mathbb{R}$ such that

- $H_1 := H$, H_2, …, H_N are functionally independent, and
- H_1, …, H_N are (pairwise) in involution.

The *Liouville-Arnold theorem* states two important facts about completely integrable systems. On the computational side it says that any such system is *solvable in quadratures*, which means that a solution can by obtained by algebraic manipulations and integration. On the abstract side it determines the topology of the *symplectic leaves* of a completely integrable system. These are the sets of all points that can be reached from a given point by following the flows of H_1, …, H_N. If the symplectic leaves are compact, the Liouville-Arnold theorem states that they must be tori. Otherwise, they are cylinders.

A more general description of Hamiltonian mechanics is obtained by defining a Poisson bracket as a bilinear operation that is anti-symmetric and satisfies the *Jacobi identity*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Then, after fixing a Poisson bracket, generally different from the one given in Equation (1.4), we can define the flow of a Hamiltonian H by Equation (1.5). Such generalized Poisson brackets can also be defined in infinite dimensional spaces. The study of those systems is called *Hamiltonian field theory.*

Integrable equations rarely occur by themselves. Indeed, if a system is completely integrable in the sense of Liouville-Arnold, then not only H is the Hamiltonian of an integrable flow, but so are H_2, …, H_N. Infinite dimensional integrable equations are usually part of infinite hierarchies. A well-known example of such an integrable hierarchy is the Korteweg-de Vries (KdV) hierarchy, which will be the main example in this work.

Lax equations

A second property that is often used to characterize integrable systems is the existence of a *Lax pair.* This is a pair of operators (L, A) such that the *Lax equation*

$$\dot{L} = [A, L] := AL - LA \tag{1.6}$$

is equivalent to the given system. The fundamental property of Lax equations is that the time-evolution given by Equation (1.6) of an operator L is isospectral.

As a very simple example, consider the harmonic oscillator described by the equations

$$\dot{q} = p \qquad \text{and} \qquad \dot{p} = -\omega^2 q. \tag{1.7}$$

A possible Lax pair for this system is formed by the (2×2)-matrices

$$L := \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix} \qquad \text{and} \qquad A := \frac{1}{2}\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

Indeed, we have

$$[A, L] = \frac{1}{2}\begin{pmatrix} -\omega^2 q & \omega p \\ \omega p & \omega^2 q \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \omega^2 q & -\omega p \\ -\omega p & -\omega^2 q \end{pmatrix} = \begin{pmatrix} -\omega^2 q & \omega p \\ \omega p & \omega^2 q \end{pmatrix},$$

so the Lax equation (1.6) is equivalent to the equations of motion (1.7). The eigenvalues of L are $\lambda = \pm\sqrt{p^2 + \omega^2 q^2}$, so in this case the isospectrality of the flow is equivalent to the conservation of the energy $\frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$.

Literature

More about Hamiltonian mechanics can be found in Arnold's book [2]. For a thorough introduction to the classical theory of integrable systems, see e.g. Babelon, Bernard and Talon [3].

1.3. Variational structure of integrable hierarchies

This idea is, doubtless, rather inventive (not to say exotic) in the framework of the classical calculus of variations.

From the introduction of [5]

Even though Hamiltonian and Lagrangian mechanics are very closely related, there is no well-developed Lagrangian theory of integrable hierarchies. This thesis explores a candidate for such a notion.

Suppose we have $N - 1$ two-dimensional Lagrangian problems for a field $u(x,t)$. We think of x as a space coordinate and of t as a time coordinate. We let the Lagrangian problems share the space coordinate x, but introduce separate time coordinates t_i, $i = 2, \ldots, N$, for each of them. Simultaneous solutions can then be considered as a function $u : \mathbb{R}^N \to \mathbb{R} : (x, t_2, \ldots, t_N) \mapsto u(x, t_2, \ldots, t_N)$. The usual Lagrangian formulation of the individual equations requires the action $\int_S \mathcal{L}$ to be critical on any (x, t_i)-plane S. In the *pluri-Lagrangian* formulation, we will impose a much stronger condition: the action must be critical on any two-dimensional surface S in $\mathbb{R}^N$.

This approach might seem unmotivated, but closely related ideas are already established in other areas of mathematics. Consider for example the theory of *pluriharmonic* *functions* [9, Section 2.2]. A function of several complex variables $f : \mathbb{C}^N \to \mathbb{R}$ is pluriharmonic if

$$\frac{\partial^2 f}{\partial z_i \partial \overline{z_j}} = 0$$

for all i, j. This is equivalent to saying that $f \circ \Gamma$ is harmonic,

$$\frac{\partial^2 (f \circ \Gamma)}{\partial z \partial \overline{z}} = 0,$$

for any holomorphic curve $\Gamma : \mathbb{C} \to \mathbb{C}^N$ [12]. Another way to put this is to say that f minimizes the *Dirichlet functional*

$$\int_\Gamma \left| \frac{\partial (f \circ \Gamma)}{\partial z} \right|^2 dz \wedge d\bar{z}$$

along any holomorphic curve $\Gamma : \mathbb{C} \to \mathbb{C}^m$. This property is similar to the pluri-Lagrangian condition. Other precursors of the pluri-Lagrangian approach are *Baxter's Z-invariance* in statistical mechanics and the classical notion of *variational symmetries*, as explained in [16].

The aim of this thesis is to develop the theory of pluri-Lagrangian systems. There are three main goals. First we will derive the *multi-time Euler-Lagrange equations* for one- and two-dimensional pluri-Lagrangian systems. Then we will construct a pluri-Lagrangian form for the KdV hierarchy. Finally, we provide a brief discussion of the relation between pluri-Lagrangian systems and the established Hamiltonian theory of integrable hierarchies.

Literature

The pluri-Lagrangian approach for lattice systems is discussed by Boll, Petrera, and Suris [5]. First steps towards a pluri-Lagrangian theory for continuous integrable hierarchies were made by Suris [15, 16].

2. Pluri-Lagrangian systems

Consider a vector bundle $X = \mathbb{R}^N \times \mathbb{R}^M$ over $\mathbb{R}^N$ and its n-th jet bundle $J^n X$. Let $\mathcal{L} : J^n X \to \Omega^d(\mathbb{R}^N)$ be a smooth a form-valued function ($d < N$). We call $\mathbb{R}^N$ the *multi-time*, u the *field*, and $\mathcal{L}$ the *Lagrangian d-form*. We will use coordinates $(t_1, \ldots, t_N)$ on $\mathbb{R}^N$.

Definition 2.1. We say that the field u solves the *pluri-Lagrangian problem* for $\mathcal{L}$ if u is a critical point of the action $\int_S \mathcal{L}$ for all d-dimensional surfaces S in $\mathbb{R}^N$ simultaneously. The differential equations describing this condition are called the *multi-time Euler-Lagrange equations.* We say that they form a *pluri-Lagrangian system* and that $\mathcal{L}$ is a *pluri-Lagrangian structure* for for these equations.

To derive the multi-time Euler-Lagrange equations, we consider the vertical derivative $\delta\mathcal{L}$ of $\mathcal{L}$ in the variational bicomplex, and a generalized vertical vector field $\mathscr{V}$. The criticality condition is described by the equation

$$\int_S \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} = 0, \tag{2.1}$$

for every d-dimensional surface S and any variation $\mathscr{V}$ on the interior of S.

Proposition 2.2. *The exterior derivative* $\mathrm{d}\mathcal{L}$ *of the Lagrangian is constant on solutions of the multi-time Euler-Lagrange equations.*

Proof. Consider a solution u and a small $(d+1)$-dimensional ball B. Because $S := \partial B$ has no boundary Equation (2.1) is satisfied for any variation $\mathscr{V}$. Using the properties that $\delta\mathrm{d} + \mathrm{d}\delta = 0$ and $\iota_{\mathrm{pr}\,\mathscr{V}}\,\mathrm{d} + \mathrm{d}\,\iota_{\mathrm{pr}\,\mathscr{V}} = 0$ (Propositions A.1 and A.3 in Appendix A.1), and Stokes' theorem, we find that

$$\int_B \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta(\mathrm{d}\mathcal{L}) = -\int_B \iota_{\mathrm{pr}\,\mathscr{V}}\, \mathrm{d}(\delta\mathcal{L}) = \int_B \mathrm{d}(\iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L}) = \int_S \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} = 0.$$

Since this holds for any ball B it follows that $\iota_{\mathrm{pr}\,\mathscr{V}}\, \delta(\mathrm{d}\mathcal{L}) = 0$ for any variation $\mathscr{V}$ of a solution u. Therefore $\mathrm{d}\mathcal{L}$ is constant on solutions. □

We will take a closer look at this property in Chapter 6, when we discuss the link with Hamiltonian theory. The rest of the current chapter is devoted to the derivation of the multi-time Euler-Lagrange equations for one- and two-dimensional systems in full generality. Previously, they where only known in some special cases [15, 16].

2.1. Approximation by stepped surfaces

The key to deriving the multi-time Euler-Lagrange equations is to observe that it suffices to consider a very specific type of surface.

Definition 2.3. A d-dimensional *coordinate surface* is a surface S such that for d distinct $i_1, \dots, i_d$ and for all $x \in S$ we have $T_x S = \operatorname{span}\big(\frac{\partial}{\partial t_{i_1}}, \dots, \frac{\partial}{\partial t_{i_d}}\big)$, i.e. a surface of the form $\{(t_1, \dots, t_N) \mid \forall j \neq i_1, \dots, i_d : t_j = c_j\}$.

A *stepped surface* is a finite union of coordinate surfaces.

Lemma 2.4. *If the action is stationary on every stepped surface, then it is stationary on every smooth surface.*

The proof of this lemma is quite long and technical. The reader will not be judged for skipping it on the first reading.

Proof. Assume that the action is stationary on all d-dimensional stepped surfaces in $\mathbb{R}^N$. Let S be a smooth d-dimensional surface. Partition the space $\mathbb{R}^N$ into hypercubes C_i of edge length ε. We can choose this partitioning in such a way that the surface S does not contain the center of any of the hypercubes. Denote by $S_i^N := S \cap C_i$ the piece of S that lies in the cube C_i.

We give each hypercube its own Euclidean coordinate system $[-1,1]^N \to C_i$ and identify the hypercube with its coordinates. In each punctured hypercube $[-1,1]^N \setminus \{0\}$ we define a family of *balloon maps*

$$\mathcal{B}_\alpha^N : [-1,1]^N \setminus \{0\} \to [-1,1]^N \setminus \{0\} : x \mapsto \begin{cases} \dfrac{\alpha x}{\|x\|_{\max}} & \text{if } \|x\|_{\max} < \alpha \\ x & \text{if } \|x\|_{\max} \geq \alpha \end{cases}$$

for $\alpha \in [0,1]$. Here, $\|x\|_{\max} := \max(|x_1|, \dots |x_N|)$ denotes the maximum norm with respect to the local coordinates. The idea is that from the center of each hypercube we inflate a "square" balloon which pushes the curve away from the center until it lies on the boundary of the hypercube.

Indeed, the deformed curve $S_i^{N-1} := \mathcal{B}_1^N(S_i^N)$ lies on the boundary of the hypercube, i.e. within the $(N-1)$-faces of the hypercube. We want it to lie within the d-faces of the hypercube, which would imply that it is a stepped surface. To achieve this, we introduce a balloon map

$$\mathcal{B}_\alpha^{N-1,j} : [-1,1]^{N-1} \setminus \{0\} \to [-1,1]^{N-1} \setminus \{0\} : x \mapsto \begin{cases} \dfrac{\alpha x}{\|x\|_{\max}} & \text{if } \|x\|_{\max} < \alpha \\ x & \text{if } \|x\|_{\max} \geq \alpha \end{cases}$$

in each of the $(N-1)$-faces C_i^j of the hypercube C_i, which pushes the surface into the $(N-2)$-faces. We denote the surface we obtain this way by S_i^{N-2}. If the surface S_i^{N-1} happens to contain the center of a $(N-1)$-face, we can slightly perturb the surface

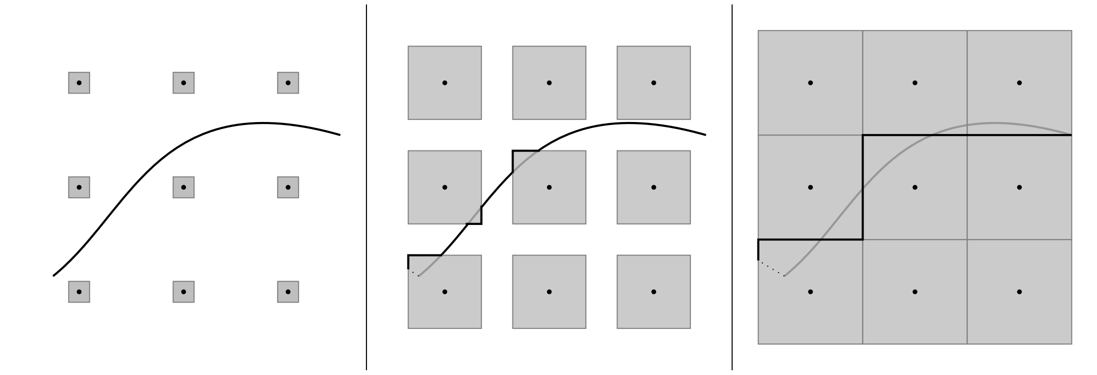

Figure 2.1.: *Balloon maps in nine adjacent squares deforming a curve in $\mathbb{R}^2$. From left to right: $\alpha = 0.2$, $\alpha = 0.7$, and $\alpha = 1$.*

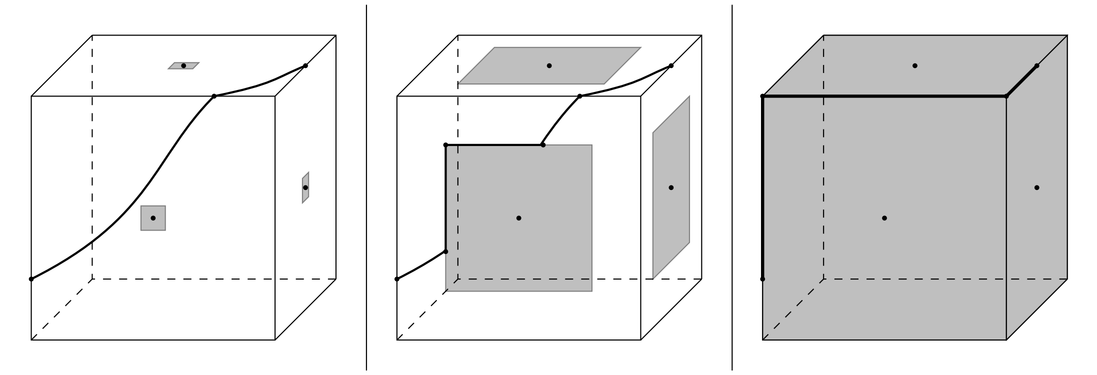

Figure 2.2.: *The second and last iteration for a curve in $\mathbb{R}^3$. From left to right: $\alpha = 0.1$, $\alpha = 0.6$, and $\alpha = 1$.*

without affecting the argument. By iterating this procedure, using balloon maps $\mathcal{B}_\alpha^{k,j}$ in each k-face C_i^j $(N \geq k \geq d+1)$, we obtain a surface S_i^d that lies in the d-faces.

Consider the $(d+1)$-dimensional surface

$$M_i := \bigcup_{k=d+1}^{N} \bigcup_{\substack{j:\ C_i^j \text{ is a} \\ k\text{-face of } C_i}} \bigcup_{\alpha \in [0,1]} \mathcal{B}_\alpha^{k,j}(S_i^k \cap C_i^j)$$

that is swept out by the consecutive application of the balloon maps to $S_i^N = S \cap C_i$. Assuming that ε is small compared to the curvature of S, the $(d+1)$-dimensional volume of each of the $\bigcup_{\alpha\in[0,1]} \mathcal{B}_\alpha^{k,j}(S_i^k \cap C_i^j)$ is of the order ε^{d+1}. The number of such volumes making up M_i only depends on the dimensions N and d, not on ε, so the

$(d+1)$-dimensional volume $|M_i|$ of M_i is of the order $|M_i| = \mathcal{O}(\varepsilon^{d+1})$.

Now consider a variation $\mathscr{V}$ with compact support and restrict the surface S to this support. Denote by $\widehat{S} := \bigcup_i S_i^d$ the stepped surface obtained from S by repeated application of balloon maps in all the hypercubes, and by $M := \bigcup_i M_i$ the $(d+1)$-dimensional surface swept out by these balloon maps. The boundary of M consists of S, $\widehat{S}$, and a small strip of area $\mathcal{O}(\varepsilon)$ connecting the boundaries of S and $\widehat{S}$ (the dotted line in Figure 2.1). The number of hypercubes intersecting S is of order ε^{-d}, so $|M| = \mathcal{O}(\varepsilon^{-d})\mathcal{O}(\varepsilon^{d+1}) = \mathcal{O}(\varepsilon)$. It follows that

$$\left| \int_{\widehat{S}} \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} - \int_S \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} \right| = \left| \int_{\partial M} \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} \right| + \mathcal{O}(\varepsilon) = \left| \int_M \mathrm{d}(\iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L}) \right| + \mathcal{O}(\varepsilon) \to 0$$

as $\varepsilon \to 0$. By assumption $\int_{\widehat{S}} \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} = 0$ for all ε, so the action on S will be stationary as well. □

2.2. Multi-time Euler-Lagrange equations for curves

Our main result is the derivation of the multi-time Euler-Lagrange equations for two-dimensional surfaces ($d = 2$). That will allow us to study the KdV hierarchy as a pluri-Lagrangian system. However, it is instructive to first derive the multi-time Euler-Lagrange equations for curves ($d = 1$).

Theorem 2.5. *Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i \,\mathrm{d}t_i$. The corresponding multi-time Euler-Lagrange equations are*

$$\frac{\delta_i L_i}{\delta u_I} = 0 \qquad \forall I \not\ni i, \tag{2.2}$$

$$\frac{\delta_i L_i}{\delta u_{Ii}} = \frac{\delta_j L_j}{\delta u_{Ij}} \qquad \forall I, \tag{2.3}$$

where i and j are distinct, and

$$\frac{\delta_i L_i}{\delta u_I} := \sum_{\alpha \in \mathbb{N}} (-1)^\alpha \,\mathrm{D}_i^\alpha \frac{\partial L_i}{\partial u_{Ii^\alpha}} = \frac{\partial L_i}{\partial u_I} - \mathrm{D}_i \frac{\partial L_i}{\partial u_{Ii}} + \mathrm{D}_i^2 \frac{\partial L_i}{\partial u_{Ii^2}} - \dots$$

is the variational derivative of L_i with respect to u_I.

Remark. Note that $\frac{\delta_i}{\delta u_I}$ differs from the traditional variational derivative in that the additional derivatives are with respect to the coordinate t_i only. If it is clear in which direction the extra derivatives are taken, we will also use the notation $\frac{\delta}{\delta u_I}$.

Remark. In the special case that $\mathcal{L}$ only depends on the first jet bundle, the system (2.2)–(2.3) reduces to the equations found in [15].

Proof of Theorem 2.5. It is sufficient to look at a general L-shaped curve $S = S_i \cup S_j$,

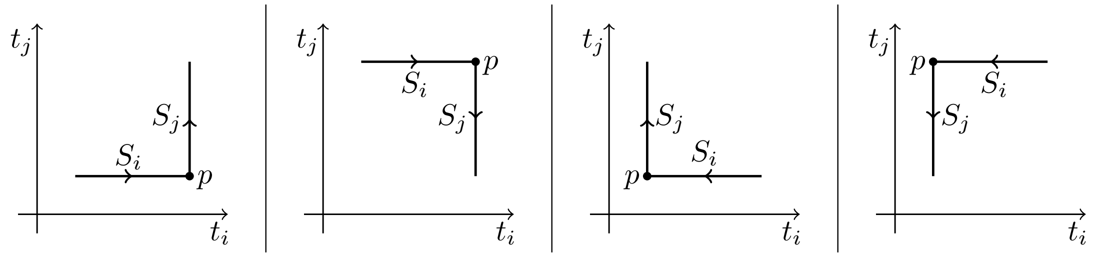

Figure 2.3.: *The four rotations of an L-shaped curve.*

where $T_x S_i = \operatorname{span}\big(\frac{\partial}{\partial t_i}\big)$ and $T_x S_j = \operatorname{span}\big(\frac{\partial}{\partial t_j}\big)$. Denote the vertex by $p := S_i \cap S_j$. We orient the curve such that S_i induces the positive orientation on the point p and S_j the negative orientation. Figure 2.3 shows this for the four possible rotations of the L-shape in the (t_i, t_j)-plane.

The variation of the action is

$$\begin{aligned}\int_S \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} &= \int_{S_i} (\iota_{\mathrm{pr}\,\mathscr{V}}\, \delta L_i)\,\mathrm{d}t_i + \int_{S_j} (\iota_{\mathrm{pr}\,\mathscr{V}}\, \delta L_j)\,\mathrm{d}t_j \\ &= \int_{S_i} \sum_I \frac{\partial L_i}{\partial u_I} \delta u_I(\mathscr{V})\,\mathrm{d}t_i + \int_{S_j} \sum_I \frac{\partial L_j}{\partial u_I} \delta u_I(\mathscr{V})\,\mathrm{d}t_j,\end{aligned}$$

where the sums are taken over all multi-indeces, including the empty one. Note that these sums are actually finite. Indeed, since $\mathcal{L}$ depends on the n-th jet bundle, all terms with $|I| > n$ vanish.

Now we expand the sum in the first of the integrals and perform integration by parts.

$$\begin{aligned}&\int_{S_i} (\iota_{\mathrm{pr}\,\mathscr{V}}\, \delta L_i)\,\mathrm{d}t_i \\ &= \int_{S_i} \sum_{I \not\ni i} \left(\frac{\partial L_i}{\partial u_I} \delta u_I(\mathscr{V}) + \frac{\partial L_i}{\partial u_{Ii}} \delta u_{Ii}(\mathscr{V}) + \frac{\partial L_i}{\partial u_{Ii^2}} \delta u_{Ii^2}(\mathscr{V}) + \frac{\partial L_i}{\partial u_{Ii^3}} \delta u_{Ii^3}(\mathscr{V}) + \ldots \right) \mathrm{d}t_i \\ &= \int_{S_i} \sum_{I \not\ni i} \left(\frac{\partial L_i}{\partial u_I} - \mathrm{D}_i \frac{\partial L_i}{\partial u_{Ii}} + \mathrm{D}_i^2 \frac{\partial L_i}{\partial u_{Ii^2}} - \mathrm{D}_i^3 \frac{\partial L_i}{\partial u_{Ii^3}} + \ldots \right) \delta u_I(\mathscr{V})\,\mathrm{d}t_i \\ &\quad + \sum_{I \not\ni i} \left(\frac{\partial L_i}{\partial u_{Ii}} \delta u_I(\mathscr{V}) + \left(\frac{\partial L_i}{\partial u_{Ii^2}} \delta u_{Ii}(\mathscr{V}) - \mathrm{D}_i \frac{\partial L_i}{\partial u_{Ii^2}} \delta u_I(\mathscr{V}) \right) \right. \\ &\qquad \left. \left. + \left(\frac{\partial L_i}{\partial u_{Ii^3}} \delta u_{Ii^2}(\mathscr{V}) - \mathrm{D}_i \frac{\partial L_i}{\partial u_{Ii^3}} (\mathscr{V}) \delta u_{Ii}(\mathscr{V}) + \mathrm{D}_i^2 \frac{\partial L_i}{\partial u_{Ii^3}} \delta u_I(\mathscr{V}) \right) + \ldots \right) \right|_p .\end{aligned}$$

Using the language of variational derivatives, this reads

$$\int_{S_i} (\iota_{\mathrm{pr}\,\mathscr{V}}\, \delta L_i)\,\mathrm{d}t_i = \int_{S_i} \sum_{I \not\ni i} \frac{\delta_i L_i}{\delta u_I} \delta u_I(\mathscr{V})\,\mathrm{d}t_i + \sum_{I \not\ni i} \left(\frac{\delta_i L_i}{\delta u_{Ii}} \delta u_I(\mathscr{V}) + \frac{\delta_i L_i}{\delta u_{Ii^2}} \delta u_{Ii}(\mathscr{V}) + \dots \right) \bigg|_p$$

$$= \int_{S_i} \sum_{I \not\ni i} \frac{\delta_i L_i}{\delta u_I} \delta u_I(\mathscr{V})\,\mathrm{d}t_i + \sum_{I} \left(\frac{\delta_i L_i}{\delta u_{Ii}} \delta u_I(\mathscr{V}) \right) \bigg|_p .$$

The other piece, S_j, contributes

$$\int_{S_j} (\iota_{\mathrm{pr}\,\mathscr{V}}\, \delta L_j)\,\mathrm{d}t_j = \int_{S_j} \sum_{I \not\ni j} \frac{\delta_j L_j}{\delta u_I} \delta u_I(\mathscr{V})\,\mathrm{d}t_j - \sum_{I} \left(\frac{\delta_j L_j}{\delta u_{Ij}} \delta u_I(\mathscr{V}) \right) \bigg|_p ,$$

where the minus sign comes from the fact that S_j induces negative orientation on the point p. Summing the two contributions we find

$$\int_S \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta \mathcal{L} = \int_{S_i} \sum_{I \not\ni i} \frac{\delta_i L_i}{\delta u_I} \delta u_I(\mathscr{V})\,\mathrm{d}t_i + \int_{S_j} \sum_{I \not\ni j} \frac{\delta_j L_j}{\delta u_I} \delta u_I(\mathscr{V})\,\mathrm{d}t_j + \sum_{I} \left(\frac{\delta_i L_i}{\delta u_{Ii}} - \frac{\delta_j L_j}{\delta u_{Ij}} \right) \delta u_I(\mathscr{V}) \bigg|_p . \tag{2.4}$$

Now require that the variation (2.4) of the action is zero for any variation $\mathscr{V}$. If we consider variations that are zero on S_j, then we find for every I which does not contain i that

$$\frac{\delta_i L_i}{\delta u_I} = 0.$$

Given this equation, and its analogue for the index j, only the last term remains in the right hand side of Equation (2.4). Considering variations around the vertex p we then find

$$\frac{\delta_i L_i}{\delta u_{Ii}} = \frac{\delta_j L_j}{\delta u_{Ij}}.$$

It is clear these two equations combined are also sufficient for the action to be critical. □

2.3. Euler-Lagrange equations for two-dimensional surfaces

The two-dimensional case ($d = 2$) covers many known integrable hierarchies, including the potential KdV hierarchy which we will discuss in detail later on. We consider a Lagrangian two-form $\mathcal{L} = \sum_{i<j} L_{ij}\,\mathrm{d}t_i \wedge \mathrm{d}t_j$ and we will use the notational convention $L_{ji} = -L_{ij}$.

Theorem 2.6. *The multi-time Euler-Lagrange equations for two-dimensional surfaces are*

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni i, j, \tag{2.5}$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{Ij}} = \frac{\delta_{ik} L_{ik}}{\delta u_{Ik}} \qquad \forall I \not\ni i, \tag{2.6}$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{Iij}} + \frac{\delta_{jk} L_{jk}}{\delta u_{Ijk}} + \frac{\delta_{ki} L_{ki}}{\delta u_{Iki}} = 0 \qquad \forall I, \tag{2.7}$$

where i, j, and k are distinct, and

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} := \sum_{\alpha,\beta \in \mathbb{N}} (-1)^{\alpha+\beta} \, \mathrm{D}_i^\alpha \, \mathrm{D}_j^\beta \, \frac{\partial L_{ij}}{\partial u_{I i^\alpha j^\beta}}$$

is the variational derivative.

Remark. In the special case that $\mathcal{L}$ depends on the second jet bundle and $N = 3$, the system (2.5)–(2.7) reduces to the equations given in [16].

Before proceeding with the proof of Theorem 2.6, we introduce some terminology and prove a lemma. A two-dimensional stepped surface consisting of q flat pieces intersecting at some point p is called a *q-flower* around p. The flat pieces are called its *petals*. If the action is stationary on every q-flower, then it is stationary on any stepped surface. By Lemma 2.4 the action will then be stationary on any surface. The following Lemma shows that it is sufficient to consider 3-flowers.

Lemma 2.7. *Take any $q > 3$. If the action is stationary on every 3-flower, it is stationary on every q-flower.*

Proof. Let F be a q-flower around a point p. Denote its petals corresponding to coordinate directions (t_{i_1}, t_{i_2}), (t_{i_2}, t_{i_3}), …, (t_{i_q}, t_{i_1}) by S_{12}, S_{23}, …, S_{q1} respectively. Consider the 3-flower $F_{123} = S_{12} \cup S_{23} \cup S_{31}$, where S_{31} is a petal in the coordinate direction (t_{i_3}, t_{i_1}) such that F_{123} is a flower around the same point p. Similarly, define $F_{134}, \ldots, F_{1\,q-1\,q}$. Then (for any integrand)

$$\begin{aligned}\int_{F_{123}} + \int_{F_{134}} + \ldots + \int_{F_{1\,q-1\,q}} \\ = \int_{S_{12}} + \int_{S_{23}} + \int_{S_{31}} + \int_{S_{13}} + \int_{S_{34}} + \int_{S_{41}} + \ldots + \int_{S_{1\,q-1}} + \int_{S_{q-1\,q}} + \int_{S_{q1}} .\end{aligned}$$

Here, S_{21}, S_{32}, … are the petals S_{12}, S_{23}, … but with opposite orientation (see Figure 2.4), hence all terms where the index of S contains 1 cancel, except for the first and

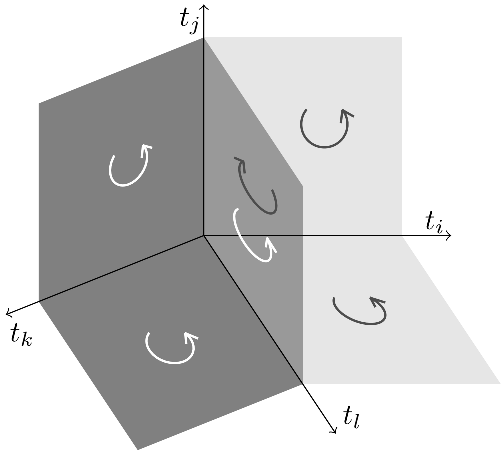

Figure 2.4.: *Two 3-flowers composed to form a 4-flower. The common petal does not contribute to the integral because it occurs twice with opposite orientation.*

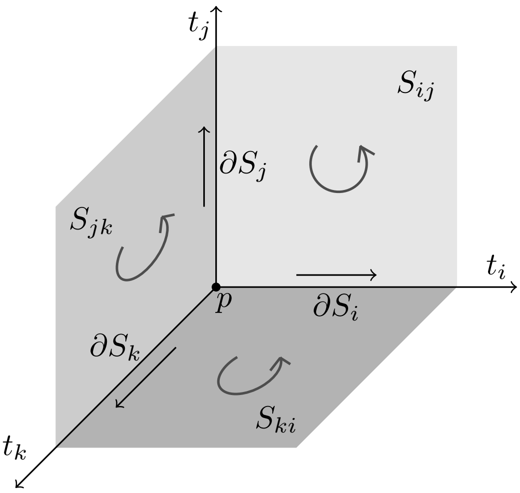

Figure 2.5.: *The 3-flower considered in the proof of Theorem 2.6.*

last, leaving

$$\int_{F_{123}} + \ldots + \int_{F_{1\,q-1\,q}} = \int_{S_{12}} + \int_{S_{23}} + \int_{S_{34}} + \ldots + \int_{S_{q-1\,q}} + \int_{S_{q1}} = \int_F .$$

By assumption the action is stationary on every 3-flower, so

$$\int_F \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} = \int_{F_{123}} \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} + \ldots + \int_{F_{1\,q-1\,q}} \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} = 0$$ □

Proof of Theorem 2.6. Consider a 3-flower $S = S_{ij} \cup S_{jk} \cup S_{ki}$ around the point p. Denote its interior edges by

$$\partial S_i := S_{ij} \cap S_{ki}, \qquad \partial S_j := S_{jk} \cap S_{ij}, \qquad \partial S_k := S_{ki} \cap S_{jk}.$$

On S_i, S_j and S_k we choose the orientations that induce negative orientation on p. We choose the orientation on the petals in such a way that the orientations of S_i, S_j and S_k are induced by S_{ij}, S_{jk} and S_{ki} respectively. Then the orientations of S_i, S_j and S_k are the opposite of those induced by S_{ki}, S_{ij} and S_{jk} respectively (see Figure 2.5).

We will calculate

$$\int_S \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} = \int_{S_{ij}} \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} + \int_{S_{jk}} \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} + \int_{S_{ki}} \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} \tag{2.8}$$

and require it to be zero for any variation $\mathscr{V}$ which vanishes on the (outer) boundary of S. This will give us the multi-time Euler-Lagrange equations.

For the first term of the right hand side of Equation (2.8) we find

$$\begin{aligned}\int_{S_{ij}} \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} &= \int_{S_{ij}} (\iota_{\mathrm{pr}\,\mathscr{V}}\, \delta L_{ij})\, \mathrm{d}t_i \wedge \mathrm{d}t_j \\ &= \int_{S_{ij}} \sum_I \frac{\partial L_{ij}}{\partial u_I} \delta u_I(\mathscr{V})\, \mathrm{d}t_i \wedge \mathrm{d}t_j \\ &= \int_{S_{ij}} \sum_{I \not\ni i,j} \sum_{\lambda,\mu \in \mathbb{N}} \frac{\partial L_{ij}}{\partial u_{Ii^\lambda j^\mu}} \delta u_{Ii^\lambda j^\mu}(\mathscr{V})\, \mathrm{d}t_i \wedge \mathrm{d}t_j.\end{aligned}$$

First we perform integration by parts with respect to t_i as many times as possible.

$$\begin{aligned}\int_{S_{ij}} \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} = &\int_{S_{ij}} \sum_{I \not\ni i,j} \sum_{\lambda,\mu \in \mathbb{N}} (-1)^\lambda\, \mathrm{D}_i^\lambda \frac{\partial L_{ij}}{\partial u_{Ii^\lambda j^\mu}} \delta u_{Ij^\mu}(\mathscr{V})\, \mathrm{d}t_i \wedge \mathrm{d}t_j. \\ &- \int_{\partial S_j} \sum_{I \not\ni i,j} \sum_{\lambda,\mu \in \mathbb{N}} \sum_{\pi=0}^{\lambda-1} (-1)^\pi\, \mathrm{D}_i^\pi \frac{\partial L_{ij}}{\partial u_{Ii^\lambda j^\mu}} \delta u_{Ii^{\lambda-\pi-1} j^\mu}(\mathscr{V})\, \mathrm{d}t_j,\end{aligned}$$

where the minus sign is due to the choice of orientations. Next integrate by parts with respect to t_j as many times as possible.

$$\int_{S_{ij}} \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} = \int_{S_{ij}} \sum_{I \not\ni i,j} \sum_{\lambda,\mu \in \mathbb{N}} (-1)^{\lambda+\mu}\, \mathrm{D}_i^\lambda\, \mathrm{D}_j^\mu \frac{\partial L_{ij}}{\partial u_{Ii^\lambda j^\mu}} \delta u_I(\mathscr{V})\, \mathrm{d}t_i \wedge \mathrm{d}t_j. \tag{2.9}$$

$$- \int_{\partial S_j} \sum_{I \not\ni i,j} \sum_{\lambda,\mu \in \mathbb{N}} \sum_{\pi=0}^{\lambda-1} (-1)^\pi\, \mathrm{D}_i^\pi \frac{\partial L_{ij}}{\partial u_{Ii^\lambda j^\mu}} \delta u_{Ii^{\lambda-\pi-1} j^\mu}(\mathscr{V})\, \mathrm{d}t_j \tag{2.10}$$

$$- \int_{\partial S_i} \sum_{I \not\ni i,j} \sum_{\lambda,\mu \in \mathbb{N}} \sum_{\rho=0}^{\mu-1} (-1)^{\lambda+\rho}\, \mathrm{D}_i^\lambda\, \mathrm{D}_j^\rho \frac{\partial L_{ij}}{\partial u_{Ii^\lambda j^\mu}} \delta u_{Ij^{\mu-\rho-1}}(\mathscr{V})\, \mathrm{d}t_i. \tag{2.11}$$

The minus sign of (2.11) is not due to orientation, but due to the anti-symmetry of two-forms: for any function f there holds $\mathrm{d}(f\,\mathrm{d}t_i) = (\mathrm{D}_j f)\,\mathrm{d}t_j \wedge \mathrm{d}t_i = -(\mathrm{D}_j f)\,\mathrm{d}t_i \wedge \mathrm{d}t_j$.

We can rewrite the first integral (2.9) as

$$\int_{S_{ij}} \sum_{I \not\ni i,j} \frac{\delta_{ij} L_{ij}}{\delta u_I} \delta u_I(\mathscr{V})\, \mathrm{d}t_i \wedge \mathrm{d}t_j.$$

The last integral (2.11) takes a similar form if we replace the index μ by $\beta = \mu - \rho - 1$.

$$\begin{aligned}
-\int_{\partial S_i} \sum_{I \not\ni i,j} \sum_{\lambda,\mu \in \mathbb{N}} \sum_{\rho=0}^{\mu-1} (-1)^{\lambda+\rho} \mathrm{D}_i^\lambda \mathrm{D}_j^\rho \frac{\partial L_{ij}}{\partial u_{Ii^\lambda j^\mu}} \delta u_{Ij^{\mu-\rho-1}}(\mathscr{V}) \,\mathrm{d}t_i \\
= -\int_{\partial S_i} \sum_{I \not\ni i,j} \sum_{\beta,\lambda,\rho \in \mathbb{N}} (-1)^{\lambda+\rho} \mathrm{D}_i^\lambda \mathrm{D}_j^\rho \frac{\partial L_{ij}}{\partial u_{Ii^\lambda j^{\beta+\rho+1}}} \delta u_{Ij^\beta}(\mathscr{V}) \,\mathrm{d}t_i \\
= -\int_{\partial S_i} \sum_{I \not\ni i,j} \sum_{\beta \in \mathbb{N}} \frac{\delta_{ij} L_{ij}}{\delta u_{Ij^{\beta+1}}} \delta u_{Ij^\beta}(\mathscr{V}) \,\mathrm{d}t_i \\
= -\int_{\partial S_i} \sum_{I \not\ni i} \frac{\delta_{ij} L_{ij}}{\delta u_{Ij}} \delta u_I(\mathscr{V}) \,\mathrm{d}t_i.
\end{aligned}$$

To write the integral (2.10) in this form we first perform integration by parts.

$$\begin{aligned}
-\int_{\partial S_j} \sum_{I \not\ni i,j} \sum_{\lambda,\mu \in \mathbb{N}} \sum_{\pi=0}^{\lambda-1} (-1)^{\pi} \mathrm{D}_i^\pi \frac{\partial L_{ij}}{\partial u_{Ii^\lambda j^\mu}} \delta u_{Ii^{\lambda-\pi-1}j^\mu}(\mathscr{V}) \,\mathrm{d}t_j \\
= -\int_{\partial S_j} \sum_{I \not\ni i,j} \sum_{\lambda,\mu \in \mathbb{N}} \sum_{\pi=0}^{\lambda-1} (-1)^{\pi+\mu} \mathrm{D}_i^\pi \mathrm{D}_j^\mu \frac{\partial L_{ij}}{\partial u_{Ii^\lambda j^\mu}} \delta u_{Ii^{\lambda-\pi-1}}(\mathscr{V}) \,\mathrm{d}t_j \\
+ \sum_{I \not\ni i,j} \sum_{\lambda,\mu \in \mathbb{N}} \sum_{\pi=0}^{\lambda-1} \sum_{\rho=0}^{\mu-1} \left((-1)^{\pi+\rho} \mathrm{D}_i^\pi \mathrm{D}_j^\rho \frac{\partial L_{ij}}{\partial u_{Ii^\lambda j^\mu}} \delta u_{Ii^{\lambda-\pi-1}j^{\mu-\rho-1}}(\mathscr{V}) \right) \Bigg|_p .
\end{aligned}$$

Then we replace λ by $\alpha = \lambda - \pi - 1$ and in the last term μ by $\beta = \mu - \rho - 1$.

$$\begin{aligned}
-\int_{\partial S_j} \sum_{I \not\ni i,j} \sum_{\lambda,\mu \in \mathbb{N}} \sum_{\pi=0}^{\lambda-1} (-1)^{\pi} \mathrm{D}_i^\pi \frac{\partial L_{ij}}{\partial u_{Ii^\lambda j^\mu}} \delta u_{Ii^{\lambda-\pi-1}j^\mu}(\mathscr{V}) \,\mathrm{d}t_j \\
= -\int_{\partial S_j} \sum_{I \not\ni i,j} \sum_{\alpha,\mu,\pi \in \mathbb{N}} (-1)^{\pi+\mu} \mathrm{D}_i^\pi \mathrm{D}_j^\mu \frac{\partial L_{ij}}{\partial u_{Ii^{\alpha+\pi+1}j^\mu}} \delta u_{Ii^\alpha}(\mathscr{V}) \,\mathrm{d}t_j \\
+ \sum_{I \not\ni i,j} \sum_{\alpha,\beta,\pi,\rho \in \mathbb{N}} \left((-1)^{\pi+\rho} \mathrm{D}_i^\pi \mathrm{D}_j^\rho \frac{\partial L_{ij}}{\partial u_{Ii^{\alpha+\pi+1}j^{\beta+\rho+1}}} \delta u_{Ii^\alpha j^\beta}(\mathscr{V}) \right) \Bigg|_p \\
= -\int_{\partial S_j} \sum_{I \not\ni i,j} \sum_{\alpha \in \mathbb{N}} \frac{\delta_{ij} L_{ij}}{\delta u_{Ii^{\alpha+1}}} \delta u_{Ii^\alpha}(\mathscr{V}) \,\mathrm{d}t_j + \sum_{I \not\ni i,j} \sum_{\alpha,\beta \in \mathbb{N}} \left(\frac{\delta_{ij} L_{ij}}{\delta u_{Ii^{\alpha+1}j^{\beta+1}}} \delta u_{Ii^\alpha j^\beta}(\mathscr{V}) \right) \Bigg|_p \\
= -\int_{\partial S_j} \sum_{I \not\ni j} \frac{\delta_{ij} L_{ij}}{\delta u_{Ii}} \delta u_I(\mathscr{V}) \,\mathrm{d}t_j + \sum_{I} \left(\frac{\delta_{ij} L_{ij}}{\delta u_{Iij}} \delta u_I(\mathscr{V}) \right) \Bigg|_p .
\end{aligned}$$

Putting everything together we find

$$\begin{aligned}\int_{S_{ij}} \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} = \int_{S_{ij}} \sum_{I \not\ni i,j} \frac{\delta_{ij} L_{ij}}{\delta u_I} \delta u_I(\mathscr{V})\, \mathrm{d}t_i \wedge \mathrm{d}t_j - \int_{\partial S_i} \sum_{I \not\ni i} \frac{\delta_{ij} L_{ij}}{\delta u_{Ij}} \delta u_I(\mathscr{V})\, \mathrm{d}t_i \\ - \int_{\partial S_j} \sum_{I \not\ni j} \frac{\delta_{ij} L_{ij}}{\delta u_{Ii}} \delta u_I(\mathscr{V})\, \mathrm{d}t_j + \sum_I \left(\frac{\delta_{ij} L_{ij}}{\delta u_{Iij}} \delta u_I(\mathscr{V}) \right) \bigg|_p .\end{aligned}$$

Expressions for the integrals over S_{jk} and S_{ki} are found by cyclic permutation of the indices. Finally, keeping in mind that $L_{ki} = -L_{ik}$, we obtain

$$\begin{aligned}\int_S \iota_{\mathrm{pr}\,\mathscr{V}}\, \delta\mathcal{L} = \int_{S_{ij}} \sum_{I \not\ni i,j} \frac{\delta_{ij} L_{ij}}{\delta u_I} \delta u_I(\mathscr{V})\, \mathrm{d}t_i \wedge \mathrm{d}t_j - \int_{\partial S_i} \sum_{I \not\ni i} \left(\frac{\delta_{ij} L_{ij}}{\delta u_{Ij}} - \frac{\delta_{ki} L_{ik}}{\delta u_{Ik}} \right) \delta u_I(\mathscr{V})\, \mathrm{d}t_i \\ + \sum_I \left(\frac{\delta_{ij} L_{ij}}{\delta u_{Iij}} \delta u_I(\mathscr{V}) \right) \bigg|_p \quad + \text{ cyclic permutations in } i, j, k. \qquad (2.12)\end{aligned}$$

From this we can read off the multi-time Euler-Lagrange equations. □

3. Pluri-Lagrangian structure of the sine-Gordon equation

The first example of a pluri-Lagrangian system in our sense was discussed by Suris [16] at a time when the multi-time Euler-Lagrange equations were only known in some special cases. It consists of the sine-Gordon equation and the modified KdV equation. The presentation in this chapter closely follows [16].

The sine-Gordon equation $u_{xy} = \sin u$ is the Euler-Lagrange equation for the Lagrangian

$$L := \frac{1}{2} u_x u_y - \cos u.$$

Consider the vector field $\varphi \frac{\partial}{\partial u}$ with

$$\varphi := u_{xxx} + \frac{1}{2} u_x^3$$

and its prolongation $\mathrm{D}_\varphi := \sum_I (\mathrm{D}_I \varphi) \frac{\partial}{\partial u_I}$. It is known that D_φ is a variational symmetry for the sine-Gordon equation [13, p. 336]. In particular, we have that

$$\mathrm{D}_\varphi L = -\mathrm{D}_x N + \mathrm{D}_y M \tag{3.1}$$

with

$$\begin{aligned} M &:= \frac{1}{2}\varphi u_x - \frac{1}{8} u_x^4 + \frac{1}{2} u_{xx}^2, \\ N &:= -\frac{1}{2}\varphi u_y + \frac{1}{2} u_x^2 \cos u + u_{xx}(u_{xy} - \sin u). \end{aligned}$$

Now we introduce a new independent variable z corresponding to the "flow" of the generalized vector field D_φ, i.e. $u_z = \varphi$. Consider simultaneous solutions of the Euler-Lagrange equation $\frac{\delta L}{\delta u} = 0$ and of the flow $u_z = \varphi$ as functions of 3 independent variables x, y, and z. Then Equation (3.1) expresses the closedness of the two-form

$$\mathcal{L} := L\, dx \wedge dy + M\, dz \wedge dx + N\, dy \wedge dz.$$

In other words, $\mathrm{d}\mathcal{L} = 0$ on simultaneous solutions, so is consistent with Proposition 2.2. Therefore $\mathcal{L}$ is a reasonable candidate for a Lagrangian two-form.

Theorem 3.1. *The multi-time Euler-Lagrange equations for the Lagrangian two-form*

$$\mathcal{L} = L_{12}\ dx \wedge dy + L_{13}\ dz \wedge dx + L_{23}\ dy \wedge dz$$

with the components

$$L_{12} = \frac{1}{2}u_x u_y - \cos u, \tag{3.2}$$

$$L_{13} = \frac{1}{2}u_x u_z - \frac{1}{8}u_x^4 + \frac{1}{2}u_{xx}^2, \tag{3.3}$$

$$L_{23} = -\frac{1}{2}u_y u_z + \frac{1}{2}u_x^2 \cos u + u_{xx}(u_{xy} - \sin u), \tag{3.4}$$

consist of the sine-Gordon equation

$$u_{xy} = \sin u,$$

the modified KdV equation

$$u_z = u_{xxx} + \frac{1}{2}u_x^3,$$

and corollaries thereof. On solutions of either of these equations the two-form $\mathcal{L}$ is closed.

Proof. Let us calculate the multi-time Euler-Lagrange equations (2.5)–(2.7) one by one:

- The equation $\dfrac{\delta_{12} L_{12}}{\delta u} = 0$ yields $$u_{xy} = \sin u.$$

  For any $\alpha > 0$ the equation $\dfrac{\delta_{12} L_{12}}{\delta u_{z^\alpha}} = 0$ yields $0 = 0$.

- The equation $\dfrac{\delta_{13} L_{13}}{\delta u} = 0$ yields $$u_{xz} = \frac{3}{2}u_x^2 u_{xx} + u_{xxxx}.$$

  For any $\alpha > 0$ the equation $\dfrac{\delta_{13} L_{13}}{\delta u_{y^\alpha}} = 0$ yields $0 = 0$.

- The equation $\dfrac{\delta_{23} L_{23}}{\delta u} = 0$ yields $$u_{yz} = \frac{1}{2}u_x^2 \sin u + u_{xx} \cos u.$$

  The equation $\dfrac{\delta_{23} L_{23}}{\delta u_x} = 0$ yields $$u_{yxx} = u_x \cos u.$$

  The equation $\dfrac{\delta_{23} L_{23}}{\delta u_{xx}} = 0$ yields $$u_{xy} = \sin u.$$

  For any $\alpha > 2$, the equation $\dfrac{\delta_{23} L_{23}}{\delta u_{x^\alpha}} = 0$ yields $0 = 0$.

- The equation $\dfrac{\delta_{13}L_{13}}{\delta u_x} = \dfrac{\delta_{23}L_{23}}{\delta u_y}$ yields $u_z = u_{xxx} + \dfrac{1}{2}u_x^3.$

  The equation $\dfrac{\delta_{13}L_{13}}{\delta u_{xx}} = \dfrac{\delta_{23}L_{23}}{\delta u_{xy}}$ yields $u_{xx} = u_{xx}$.

  For any other I the equation $\dfrac{\delta_{13}L_{13}}{\delta u_{Ix}} = \dfrac{\delta_{23}L_{23}}{\delta u_{Iy}}$ yields $0 = 0$.

- The equation $\dfrac{\delta_{12}L_{12}}{\delta u_y} = \dfrac{\delta_{13}L_{13}}{\delta u_z}$ yields $\frac{1}{2}u_x = \frac{1}{2}u_x$.

  For any nonempty I, the equation $\dfrac{\delta_{12}L_{12}}{\delta u_{Iy}} = \dfrac{\delta_{13}L_{13}}{\delta u_{Iz}}$ yields $0 = 0$.

- The equation $\dfrac{\delta_{12}L_{12}}{\delta u_x} = \dfrac{\delta_{23}L_{32}}{\delta u_z}$ yields $\frac{1}{2}u_y = \frac{1}{2}u_y$.

  For any nonempty I, the equation $\dfrac{\delta_{12}L_{12}}{\delta u_{Ix}} = \dfrac{\delta_{23}L_{32}}{\delta u_{Iz}}$ yields $0 = 0$.

- For any I the equation $\dfrac{\delta_{12}L_{12}}{\delta v_{Ixy}} + \dfrac{\delta_{23}L_{23}}{\delta v_{Iyz}} + \dfrac{\delta_{13}L_{31}}{\delta v_{Izx}} = 0$ yields $0 = 0$.

It remains to notice that all nontrivial equations in this list are corollaries of the equations $u_{xy} = \sin u$ and $u_z = u_{xxx} + \frac{1}{2}u_x^3$, derived by differentiation.

The closedness of $\mathcal{L}$ can be verified by direct calculation:

$$\begin{aligned}
\mathrm{D}_z\, L_{12} - \mathrm{D}_y\, L_{13} + \mathrm{D}_x\, L_{23} &= \frac{1}{2}(u_{yz}u_x + u_{zx}u_y) + u_z \sin u \\
&\quad - \frac{1}{2}u_{yz}u_x - \frac{1}{2}u_z u_{xy} + \frac{1}{2}u_x^3 u_{xy} - u_{xx}u_{xxy} \\
&\quad - \frac{1}{2}u_{xz}u_y - \frac{1}{2}u_z u_{xy} + u_x u_{xx}\cos u - \frac{1}{2}u_x^3 \sin u \\
&\quad + u_{xxx}(u_{xy} - \sin u) + u_{xx}(u_{xxy} - u_x \cos u) \\
&= -\left(u_z - \frac{1}{2}u_x^3 - u_{xxx}\right)(u_{xy} - \sin u). \qquad \square
\end{aligned}$$

Remark. The sine-Gordon equation and the modified KdV equation are the simplest equations of their respective hierarchies. Furthermore, those hierarchies can be seen as the positive and negative parts of one single hierarchy that is infinite in both directions [11, Sections 3c and 5k]. It seems likely that this whole hierarchy possesses a pluri-Lagrangian structure.

4. The KdV Hierarchy

Our second and most important example will be the Korteweg-de Vries (KdV) hierarchy. The KdV equation played a central role in the development of the modern theory of integrable systems. That does not only make for an interesting history lesson (see e.g. Kasman [8] for a very accessible history, or Palais [14] for a more technical text), but it also has the consequence that many different approaches are available to introduce and study this equation (see e.g. Dickey [6]). We will use Lax pairs and pseudodifferential operators because they provide an elegant way to construct the whole KdV hierarchy at once.

This chapter borrows heavily from [6] and to a lesser extent from [4].

4.1. Lax Pairs

Consider the *Schrödinger operator*

$$\mathscr{L} := \partial^2 + u. \tag{4.1}$$

We look for differential operators $\mathscr{P}$ such that the commutator $[\mathscr{P}, \mathscr{L}]$ is a zeroth-order operator, i.e a function. Then it is possible to consider the Lax equation $\mathscr{L}_t = [\mathscr{P}, \mathscr{L}]$, where the subscript t denotes the time derivative taken coefficient-wise. This equation is equivalent to the evolution equation

$$u_t = [\mathscr{P}, \mathscr{L}]. \tag{4.2}$$

The equations obtained by plugging different operators $\mathscr{P}$ into Equation (4.2) form the KdV hierarchy. To construct suitable operators $\mathscr{P}$ we need a generalization of the concept of differential operator.

Definition 4.1. A *pseudodifferential operator (PDO)* $\mathscr{A}$ is a formal series of the form

$$\mathscr{A} = \sum_{i=-\infty}^{m} A_i \partial^i, \qquad A_m \neq 0,$$

where the coefficients A_i are differential polynomials in u, u_x, u_{xx}, $\ldots$. The highest index, m, is called the *order* of $\mathscr{A}$.

The *differential part* $\mathscr{A}_+$ and the *integral part* $\mathscr{A}_-$ of $\mathscr{A}$ are defined as

$$\mathscr{A}_+ := \sum_{i=0}^{m} A_i \partial^i \qquad \text{and} \qquad \mathscr{A}_- := \sum_{i=-\infty}^{-1} A_i \partial^i.$$

The coefficient of ∂^{-1} in a PDO $\mathscr{A}$ is called the *residue* of $\mathscr{A}$ and denoted by $\operatorname{res}(\mathscr{A})$. We call $\mathscr{A}$ a *pure function* if $A_i = 0$ for all $i \neq 0$.

The set of PDOs has a ring structure. The addition of PDOs is defined coefficient-wise,

$$\sum_{i=-\infty}^{m} A_i \partial^i + \sum_{i=-\infty}^{m} B_i \partial^i = \sum_{i=-\infty}^{m} (A_i + B_i) \partial^i.$$

The multiplication of PDOs is defined by the distributivity property and the rule

$$\partial^k (A \partial^j) = A \partial^{j+k} + \binom{k}{1} (\mathrm{D}_x\, A) \partial^{j+k-1} + \binom{k}{2} (\mathrm{D}_x^2\, A) \partial^{j+k-2} + \dots,$$

where $\binom{k}{i} := \dfrac{k(k-1)\dots(k-i+1)}{i!}$ for negative values of k as well as for positive ones.

Proposition 4.2. *The PDOs $\mathscr{L}^{-1}$ and $\mathscr{L}^{\frac{1}{2}}$ exist and are unique. They commute with each other and with $\mathscr{L}$.*

Remark. In fact these properties hold for any PDO $\mathscr{A}$ with essentially the same proof. However, we will only need them for the Schrödinger operator $\mathscr{L} = \partial^2 + u$.

Proof. If $\mathscr{L}^{-1}$ exists, it must be of the form $\mathscr{L}^{-1} = \partial^{-2} + X_3 \partial^{-3} + X_4 \partial^{-4} + \dots$. The identity $\mathscr{L}\mathscr{L}^{-1} = 1$ gives

$$1 + X_3 \partial^{-1} + (X_4 + u)\, \partial^{-2} + (X_5 + u X_3)\, \partial^{-3} + (X_6 + u X_4)\, \partial^{-4} + \dots = 1,$$

from which we obtain $X_3 = 0$, $X_4 = -u$, and a recurrence relation $X_{k+2} = -u X_k$. This determines $\mathscr{L}^{-1}$ to be

$$\mathscr{L}^{-1} = \partial^{-2} - u \partial^{-4} + u^2 \partial^{-6} - \dots.$$

Similarly, if $\mathscr{L}^{\frac{1}{2}}$ exists, it must be of the form $\mathscr{L}^{\frac{1}{2}} = \partial^1 + X_0 + X_1 \partial^{-1} + X_2 \partial^{-2} + \dots$. The identity $\mathscr{L}^{\frac{1}{2}} \mathscr{L}^{\frac{1}{2}} = \mathscr{L}$ gives us another recurrence relation. We find that

$$\mathscr{L}^{\frac{1}{2}} = \partial + \frac{1}{2} u \partial^{-1} - \frac{1}{4} u_x \partial^{-2} + \left(\frac{1}{8} u_{xx} - \frac{1}{8} u^2 \right) \partial^{-3} + \left(\frac{3}{8} u u_x - \frac{1}{16} u_{xxx} \right) \partial^{-4} + \dots.$$

By the same procedure we can also uniquely determine the inverse $\mathscr{L}^{-\frac{1}{2}}$ of $\mathscr{L}^{\frac{1}{2}}$. It satisfies $\left(\mathscr{L}^{-\frac{1}{2}} \mathscr{L}^{-\frac{1}{2}} \right) \mathscr{L} = \mathscr{L}^{-\frac{1}{2}} \mathscr{L}^{-\frac{1}{2}} \mathscr{L}^{\frac{1}{2}} \mathscr{L}^{\frac{1}{2}} = 1$, so it is also the square root of $\mathscr{L}^{-1}$.

We need this operator to prove commutativity, which is now elementary:

$$\begin{aligned}\left[\mathscr{L}^{-1},\mathscr{L}^{\frac{1}{2}}\right] &= \mathscr{L}^{-\frac{1}{2}}\mathscr{L}^{-\frac{1}{2}}\mathscr{L}^{\frac{1}{2}} - \mathscr{L}^{\frac{1}{2}}\mathscr{L}^{-\frac{1}{2}}\mathscr{L}^{-\frac{1}{2}} = \mathscr{L}^{-\frac{1}{2}} - \mathscr{L}^{-\frac{1}{2}} = 0,\\ \left[\mathscr{L},\mathscr{L}^{-1}\right] &= 1-1=0,\\ \left[\mathscr{L},\mathscr{L}^{\frac{1}{2}}\right] &= \left(\mathscr{L}^{\frac{1}{2}}\right)^3 - \left(\mathscr{L}^{\frac{1}{2}}\right)^3 = 0.\end{aligned}$$

□

Proposition 4.2 allows us to define the operator $\mathscr{L}^{\frac{k}{2}}$ for every integer k. These operators commute with $\mathscr{L}$ as well. The differential parts of $\mathscr{L}^{\frac{1}{2}}$, $\mathscr{L}^{\frac{3}{2}}$, $\mathscr{L}^{\frac{5}{2}}$, ... are the operators we are looking for. Define

$$\mathscr{P}_k := 2^{2k-2}\left(\mathscr{L}^{\frac{2k-1}{2}}\right)_+. \tag{4.3}$$

The normalization factor 2^{2k-2} is chosen such that the highest order terms of the KdV equations will have coefficient 1.

Proposition 4.3. *The commutator $[\mathscr{P}_k,\mathscr{L}]$ is a pure function.*

Proof. Since $\mathscr{L}^{\frac{2k-1}{2}}$ commutes with $\mathscr{L}$ we have

$$\begin{aligned}[\mathscr{P}_k,\mathscr{L}] = 2^{2k-2}\left[\left(\mathscr{L}^{\frac{2k-1}{2}}\right)_+,\mathscr{L}\right] &= 2^{2k-2}\left[\mathscr{L}^{\frac{2k-1}{2}} - \left(\mathscr{L}^{\frac{2k-1}{2}}\right)_-,\mathscr{L}\right]\\ &= -2^{2k-2}\left[\left(\mathscr{L}^{\frac{2k-1}{2}}\right)_-,\mathscr{L}\right].\end{aligned}$$

The operator $\left(\mathscr{L}^{\frac{2k-1}{2}}\right)_-$ is of order -1 and $\mathscr{L}$ is of order 2, so the order of the commutator $\left[\left(\mathscr{L}^{\frac{2k-1}{2}}\right)_-,\mathscr{L}\right]$ is at most $((-1)+2)-1=0$. On the other hand, since both $\mathscr{P}_k$ and $\mathscr{L}$ are differential operators, $[\mathscr{P}_k,\mathscr{L}]$ has no terms of negative order. We conclude that $[\mathscr{P}_k,\mathscr{L}]$ only contains a zeroth-order term, hence it is a pure function. □

Definition 4.4. The *k-th KdV equation* ($k \geq 1$) is the evolution equation for u given by $\mathscr{L}_t = [\mathscr{P}_k,\mathscr{L}]$, where $\mathscr{L}$ is the Schrödinger operator (4.1) and $\mathscr{P}_k$ is given by Equation (4.3). The second KdV equation $\mathscr{L}_t = [\mathscr{P}_2,\mathscr{L}]$ is known as *the* KdV equation.

The first few KdV equations are

$$\begin{aligned}(k=1) &\qquad u_t = u_x,\\ (k=2) &\qquad u_t = u_{xxx} + 6uu_x,\\ (k=3) &\qquad u_t = u_{x^5} + 20u_xu_{xx} + 10uu_{xxx} + 30u^2u_x.\end{aligned}$$

Theorem 4.5. *The right hand sides of the KdV equations are derivatives,*

$$[\mathscr{P}_k,\mathscr{L}] = 2^{2k-1}\,\mathrm{D}_x \operatorname{res}\left(\mathscr{L}^{\frac{2k-1}{2}}\right).$$

Proof. We have

$$\left[\left(\mathscr{L}^{\frac{2k-1}{2}}\right)_+, \mathscr{L}\right] = \left(\left[-\left(\mathscr{L}^{\frac{2k-1}{2}}\right)_-, \mathscr{L}\right]\right)_+ = \left(\left[\partial^2 + u, \left(\mathscr{L}^{\frac{2k-1}{2}}\right)_-\right]\right)_+ .$$

Since $\left[u, \left(\mathscr{L}^{\frac{2k-1}{2}}\right)_-\right]$ has no differential part, it follows that

$$\left[\left(\mathscr{L}^{\frac{2k-1}{2}}\right)_+, \mathscr{L}\right] = \left(\left[\partial^2, \left(\mathscr{L}^{\frac{2k-1}{2}}\right)_-\right]\right)_+ = 2 \, \mathrm{D}_x \operatorname{res}\left(\mathscr{L}^{\frac{2k-1}{2}}\right). \qquad \square$$

Motivated by Theorem 4.5, we define

$$r_k := 2^{2k-1} \operatorname{res}\left(\mathscr{L}^{\frac{2k-1}{2}}\right). \tag{4.4}$$

Note that this also makes sense for nonpositive k, in particular

$$r_0 = \frac{1}{2} \operatorname{res}\left(\mathscr{L}^{\frac{-1}{2}}\right) = \frac{1}{2} \operatorname{res}\left(\partial^{-1} + \dots\right) = \frac{1}{2}.$$

The derivatives $\mathrm{D}_x \, r_k$ $(k \geq 1)$ are the right hand sides of the KdV equations.

The equations of the KdV hierarchy are not Lagrangian because the variational derivative $\frac{\delta}{\delta u}$ cannot produce terms of the form u_t (see Appendix A.2). However, if we introduce the *potential* v that satisfies $v_x = u$ and the functions $g_k[v] := r_k[v_x]$, the corresponding equations

$$v_{xt} = \mathrm{D}_x \, g_k \tag{4.5}$$

are in fact Lagrangian.

Definition 4.6. The set of Equations (4.5) for $k \geq 1$ is called the *differentiated potential Korteweg-de Vries (DPKdV) hierarchy.* The evolutionary version of this hierarchy,

$$v_t = g_k \qquad (k \geq 1), \tag{4.6}$$

is called the *potential Korteweg-de Vries (PKdV) hierarchy*

4.2. Variational relations

In this section we show that the right hand sides of the KdV hierarchy are related to each other through variational derivatives.

Lemma 4.7. *Let the vertical exterior derivative δ act coefficient-wise on PDOs. For any PDOs $\mathscr{A}$ and $\mathscr{B}$ the Leibniz rule holds, $\delta(\mathscr{A}\mathscr{B}) = (\delta\mathscr{A})\mathscr{B} + \mathscr{A}(\delta\mathscr{B})$.*

Proof. It is sufficient to show this for monomial PDOs. Since δ is an exterior derivative, the Leibniz rule holds when it is applied to differential polynomials, so

$$\begin{aligned}\delta(A\partial^i B\partial^j) &= \delta\left(A\sum_k \binom{i}{k} \mathrm{D}_x^k\, B\, \partial^{j+i-k}\right)\\ &= (\delta A)\sum_k \binom{i}{k} \mathrm{D}_x^k\, B\, \partial^{j+i-k} + A\sum_k \binom{i}{k}\delta(\mathrm{D}_x^k\, B)\, \partial^{j+i-k}.\end{aligned}$$

By Proposition A.2 in Appendix A.1 there holds that $\delta(\mathrm{D}_x^k\, B) = \mathrm{D}_x^k(\delta B)$, so we find

$$\begin{aligned}\delta(A\partial^i B\partial^j) &= (\delta A)\sum_k \binom{i}{k} \mathrm{D}_x^k\, B\, \partial^{j+i-k} + A\sum_k \binom{i}{k} \mathrm{D}_x^k(\delta B)\, \partial^{j+i-k}\\ &= (\delta A\, \partial^i)B\partial^j + A\partial^i(\delta B\, \partial^j).\end{aligned}$$

□

Lemma 4.8. *For any PDOs $\mathscr{A}$ and $\mathscr{B}$ we have*

$$\int \operatorname{res}(\mathscr{A}\mathscr{B}) = \int \operatorname{res}(\mathscr{B}\mathscr{A}),$$

where the integrals are formal, i.e. $\int$ denotes the equivalence class modulo D_x.

Proof. It is sufficient to prove this for monomials $\mathscr{A} = A\partial^i$ and $\mathscr{B} = B\partial^j$. Without loss of generality we can assume that $i \geq 0$, $j < 0$, and $i + j \geq -1$. We will show that the residue $\operatorname{res}[\mathscr{A}, \mathscr{B}]$ is a derivative. We have

$$[\mathscr{A}, \mathscr{B}] = A\sum_{k=0}^{i} \binom{i}{k} (\mathrm{D}_x^k\, B)\partial^{j+i-k} - B\sum_{k=0}^{\infty} \binom{j}{k} (\mathrm{D}_x^k\, A)\partial^{i+j-k},$$

hence

$$\operatorname{res}[\mathscr{A}, \mathscr{B}] = A\binom{i}{j+i+1}\, \mathrm{D}_x^{j+i+1}\, B - B\binom{j}{j+i+1}\, \mathrm{D}_x^{j+i+1}\, A.$$

From the definition of the binomial coefficient we see that

$$\binom{i}{j+i+1} = \frac{i(i-1)\dots(-j)}{(j+i+1)!} = (-1)^{i+j+1}\frac{j(j-1)\dots(-i)}{(j+i+1)!} = (-1)^{i+j+1}\binom{j}{j+i+1},$$

so

$$\begin{aligned}\operatorname{res}[\mathscr{A}, \mathscr{B}] &= \binom{i}{j+i+1}\left(A\, \mathrm{D}_x^{j+i+1}\, B + (-1)^{i+j} B\, \mathrm{D}_x^{j+i+1}\, A\right)\\ &= \mathrm{D}_x\left(\binom{i}{i+j+1}\sum_{k=0}^{i+j}(-1)^k\, \mathrm{D}_x^k\, A\, \mathrm{D}_x^{i+j-k}\, B\right).\end{aligned}$$

□

Proposition 4.9. *The quantities r_k defined by Equation* (4.4) *satisfy*

$$\int \delta r_k = \int (4k-2) r_{k-1} \delta u,$$

hence they are variational derivatives,

$$\frac{\delta r_k}{\delta u} = (4k-2) r_{k-1}.$$

Proof. By virtue of Lemma 4.7 we have

$$\begin{aligned}\int \delta r_k &= \int 2^{2k-1} \operatorname{res}\left(\delta \mathscr{L}^{\frac{2k-1}{2}}\right) \\ &= \int 2^{2k-1} \operatorname{res}\left(\left(\delta \mathscr{L}^{\frac{1}{2}}\right)\mathscr{L}^{\frac{2k-2}{2}} + \mathscr{L}^{\frac{1}{2}}\left(\delta \mathscr{L}^{\frac{1}{2}}\right)\mathscr{L}^{\frac{2k-3}{2}} + \ldots + \mathscr{L}^{\frac{2k-2}{2}}\left(\delta \mathscr{L}^{\frac{1}{2}}\right)\right).\end{aligned}$$

Using Lemma 4.8 we can write this as

$$\int \delta r_k = \int 2^{2k-2}(2k-1)\left(\delta(\mathscr{L}^{\frac{1}{2}})\mathscr{L}^{\frac{2k-2}{2}} + \mathscr{L}^{\frac{1}{2}}\delta(\mathscr{L}^{\frac{1}{2}})\mathscr{L}^{\frac{2k-3}{2}}\right).$$

Applying Lemma 4.7 once more we find

$$\begin{aligned}\int \delta r_k &= \int 2^{2k-2}(2k-1) \operatorname{res}\left(\delta \mathscr{L}\, \mathscr{L}^{\frac{2k-3}{2}}\right) \\ &= \int 2^{2k-2}(2k-1) \operatorname{res}\left(\delta u\, \mathscr{L}^{\frac{2k-3}{2}}\right) \\ &= \int 2(2k-1) r_{k-1} \delta u. \qquad \square\end{aligned}$$

Remark. Two other proofs of this relation can be found in [7] and a third one in [18] (see also Appendix A.3). However, all these proofs start from the resolvent equation (4.7) rather than directly from the Lax representation.

Corollary 4.10. *Set $h_k := \frac{1}{4k+2} g_{k+1}$, then g_k and h_k satisfy*

$$\frac{\delta g_k}{\delta v_x} = (4k-2)\, g_{k-1} \qquad \textit{and} \qquad \frac{\delta h_k}{\delta v_x} = g_k.$$

To show that the DPKdV equations are Lagrangian we need one more little lemma.

Lemma 4.11. *For any differential polynomial f in v, v_x, v_{xx}, ... we have*

$$\mathrm{D}_x \frac{\delta f}{\delta v_x} = \frac{\partial f}{\partial v} - \frac{\delta f}{\delta v}.$$

Proof. By direct calculation:

$$\begin{aligned}\mathrm{D}_x \frac{\delta f}{\delta v_x} &= \mathrm{D}_x \left(\frac{\partial f}{\partial v_x} - \mathrm{D}_x \frac{\partial f}{\partial v_{xx}} + \mathrm{D}_x^2 \frac{\partial f}{\partial v_{xxx}} + \dots \right) \\ &= \mathrm{D}_x \frac{\partial f}{\partial v_x} - \mathrm{D}_x^2 \frac{\partial f}{\partial v_{xx}} + \mathrm{D}_x^3 \frac{\partial f}{\partial v_{xxx}} + \dots = \frac{\partial f}{\partial v} - \frac{\delta f}{\delta v}.\end{aligned}$$ □

If f does not depend on v itself, this result becomes even simpler.

Corollary 4.12. *For any differential polynomial f in v_x, v_{xx}, v_{xxx}, ... there holds* $\mathrm{D}_x \frac{\delta f}{\delta v_x} = -\frac{\delta f}{\delta v}$.

Theorem 4.13. *The functions $L_k := \frac{1}{2} v_x v_t - h_k$, with $h_k = \frac{1}{4k+2} g_{k+1}$, are Lagrangians for the DPKdV Equations* (4.5).

Proof. Since $h_k = \frac{1}{4k+2} g_{k+1}$ does not depend on v directly, it follows from Corollary 4.12 that

$$\frac{\delta L_k}{\delta v} = -v_{tx} - \frac{\delta h_k}{\delta v} = -v_{tx} + \mathrm{D}_x \frac{\delta h_k}{\delta v_x}.$$

Then from Corollary 4.10 it follows that

$$\frac{\delta L_k}{\delta v} = -v_{tx} + \mathrm{D}_x\, g_k.$$ □

4.3. Resolvents

We now sketch an alternative way to introduce the KdV hierarchy, and derive an equation that will prove useful later on. Consider the the formal power series

$$\mathcal{T}^+ := \sum_{r \in \mathbb{Z}} 2^r z^{-r-2} \big(\mathscr{L}^{\frac{r}{2}} \big)_- \qquad \text{and} \qquad \mathcal{T}^- := \sum_{r \in \mathbb{Z}} 2^r (-z)^{-r-2} \big(\mathscr{L}^{\frac{r}{2}} \big)_-.$$

They are called *basic resolvents* because their average equals the resolvent $\big(4\mathscr{L} - z^2\big)^{-1}$. Indeed,

$$\frac{\mathcal{T}^+ + \mathcal{T}^-}{2} = \sum_{k \in \mathbb{Z}} 2^{2k} z^{-2k-2} \big(\mathscr{L}^k \big)_- = \sum_{k=-\infty}^{-1} 2^{2k} z^{-2k-2} \mathscr{L}^k,$$

so

$$\left(\frac{\mathcal{T}^+ + \mathcal{T}^-}{2} \right) \big(4\mathscr{L} - z^2 \big) = \sum_{k=-\infty}^{-1} 2^{2k+2} z^{-2k-2} \mathscr{L}^{k+1} - \sum_{k=-\infty}^{-1} 2^{2k} z^{-2k} \mathscr{L}^k = 1.$$

Proposition 4.14. *The PDOs $\big(4\mathscr{L} - z^2\big) \mathcal{T}^\pm$ and $\mathcal{T}^\pm \big(4\mathscr{L} - z^2\big)$ are differential operators.*

Proof. There holds

$$\begin{aligned}\left(\left(4\mathscr{L}-z^2\right)\mathcal{T}^{\pm}\right)_- &= \left(\left(4\mathscr{L}-z^2\right)\sum_{r\in\mathbb{Z}} 2^r(\pm z)^{-r-2}\left(\mathscr{L}^{\frac{r}{2}}\right)_-\right)_- \\ &= \left(\left(4\mathscr{L}-z^2\right)\sum_{r\in\mathbb{Z}} 2^r(\pm z)^{-r-2}\mathscr{L}^{\frac{r}{2}}\right)_- \\ &= \left(\sum_{r\in\mathbb{Z}} 2^{r+2}(\pm z)^{-r-2}\mathscr{L}^{\frac{r+2}{2}} - \sum_{r\in\mathbb{Z}} 2^r(\pm z)^{-r}\mathscr{L}^{\frac{r}{2}}\right)_- = 0.\end{aligned}$$

Analogously, $\left(\mathcal{T}^{\pm}\left(4\mathscr{L}-z^2\right)\right)_- = 0$. □

Corollary 4.15. *The basic resolvents $\mathcal{T}^{\pm}$ lie in the Kernel of the* Adler mapping, *defined as*

$$H:\mathcal{T}\mapsto \left(\left(4\mathscr{L}-z^2\right)\mathcal{T}\right)_+\left(4\mathscr{L}-z^2\right) - \left(4\mathscr{L}-z^2\right)\left(\mathcal{T}\left(4\mathscr{L}-z^2\right)\right)_+$$

Proof. We have

$$H(\mathcal{T}^{\pm}) = -\left(\left(4\mathscr{L}-z^2\right)\mathcal{T}^{\pm}\right)_-\left(4\mathscr{L}-z^2\right) + \left(4\mathscr{L}-z^2\right)\left(\mathcal{T}^{\pm}\left(4\mathscr{L}-z^2\right)\right)_- = 0. \quad \square$$

This corollary leads us to a useful equation that the basic resolvents $\mathcal{T}^{\pm}$ satisfy.

Theorem 4.16. *If $\mathcal{T} = T_1\partial^{-1} + T_2\partial^{-2} + \dots$ satisfies the condition $H(\mathcal{T}) = 0$, then*

$$u_x T_1 + 2\left(u - \frac{1}{4}z^2\right)\mathrm{D}_x T_1 + \frac{1}{2}D_x^3 T_1 = 0. \tag{4.7}$$

Proof. In this proof we denote total x-derivatives by a prime, $'$. The equation $H(\mathcal{T}) = 0$ is equivalent to

$$\begin{aligned}0 &= \Big(\left(\partial^2+u-\tfrac14 z^2\right)\left(T_1\partial^{-1}+T_2\partial^{-2}+\dots\right)\Big)_+\left(\partial^2+u-\tfrac14 z^2\right) \\ &\quad - \left(\partial^2+u-\tfrac14 z^2\right)\Big(\left(T_1\partial^{-1}+T_2\partial^{-2}+\dots\right)\left(\partial^2+u-\tfrac14 z^2\right)\Big)_+ \\ &= \left(T_1\partial + 2T_1' + T_2\right)\left(\partial^2+u-\tfrac14 z^2\right) - \left(\partial^2+u-\tfrac14 z^2\right)\left(T_1\partial+T_2\right) \\ &= T_1\partial^3 + \left(u-\tfrac14 z^2\right)T_1\partial + u'T_1 + 2T_1'\partial^2 + 2\left(u-\tfrac14 z^2\right)T_1' + T_2\partial^2 + \left(u-\tfrac14 z^2\right)T_2 \\ &\quad - \Big(T_1\partial^3 + 2T_1'\partial^2 + T_1''\partial + T_2\partial^2 + 2T_2'\partial + T_2'' + \left(u-\tfrac14 z^2\right)T_1\partial + \left(u-\tfrac14 z^2\right)T_2\Big) \\ &= -\left(T_1''+2T_2'\right)\partial + \left(u'T_1 + 2\left(u-\tfrac14 z^2\right)T_1' - T_2''\right).\end{aligned}$$

It follows that $T_2' = \frac{-1}{2}T_1''$ and $u'T_1 + 2\left(u-\frac14 z^2\right)T_1' + \frac12 T_1''' = 0$. □

We call Equation (4.7) the *resolvent equation*. It can be used as the starting point to define the KdV hierarchy. To see this, consider the formal power series

$$\begin{aligned}\sum_{r\in\mathbb{N}} 2^{r-1}\left[\left(\mathscr{L}^{\frac{r}{2}}\right)_+,\mathscr{L}\right](\pm z)^{-r-2} &= \sum_{r\in\mathbb{Z}} 2^{r-1}\left[\left(\mathscr{L}^{\frac{r}{2}}\right)_+,\mathscr{L}\right](\pm z)^{-r-2}\\ &= \sum_{r\in\mathbb{Z}} 2^{r-1}\left[\mathscr{L},(\mathscr{L}^{\frac{r}{2}})_-\right](\pm z)^{-r-2}\\ &= \frac{1}{2}\left[\mathscr{L},\mathcal{T}^{\pm}\right]\end{aligned}$$

Since each of the commutators $\left[\left(\mathscr{L}^{\frac{r}{2}}\right)_+,\mathscr{L}\right]$ is a pure function, we know that the only nonzero term of $[\mathscr{L},\mathcal{T}^{\pm}] = \left[\partial^2+u,\mathcal{T}^{\pm}\right]$ is the zeroth-order term, which equals $2\,\mathrm{D}_x(\operatorname{res}\mathcal{T}^{\pm})$. Therefore

$$\sum_{r\in\mathbb{N}} 2^{r-1}\left[\left(\mathscr{L}^{\frac{r}{2}}\right)_+,\mathscr{L}\right](\pm z)^{-r-2} = \mathrm{D}_x(\operatorname{res}\mathcal{T}^{\pm}).$$

On the other hand, $T_1 := \operatorname{res}\mathcal{T}^{\pm}$ satisfies the resolvent equation (4.7). This gives us the following recipe to cook up the KdV hierarchy. Take an "appropriate" solution of Equation (4.7) and expand its derivative as a power series in z. The coefficients of the odd powers in this series will be the right hand sides $[\mathscr{P}_k,\mathscr{L}] = \left[2^{2k-2}\left(\mathscr{L}^{\frac{2k-1}{2}}\right)_+,\mathscr{L}\right]$ of the KdV hierarchy. More details can be found in Appendix A.3 or in [6, Section 3.7].

5. Pluri-Lagrangian structure of the PKdV hierarchy

In the last chapter we have merely reviewed known facts about the KdV hierarchy. In this chapter we will use those facts to give a pluri-Lagrangian form of the PKdV hierarchy.

5.1. Time derivatives of the Lagrangians

In the pluri-Lagrangian formulation $\mathrm{d}\mathcal{L}$ is constant on solutions. In order to find a pluri-Lagrangian structure for the PKdV hierarchy, we will impose the stronger condition that the Lagrangian two-form is closed, $\mathrm{d}\mathcal{L} = 0$. Some of the coefficients of $\mathcal{L}$ will be the Lagrangians we computed in the last chapter. The closedness condition involves time derivatives of these coefficients and the following lemma will help in handling them.

Lemma 5.1. (a) *The polynomials*

$$a_{ij} := v_{t_j} \frac{\delta_1 h_i}{\delta v_x} + v_{xt_j} \frac{\delta_1 h_i}{\delta v_{xx}} + v_{xxt_j} \frac{\delta_1 h_i}{\delta v_{xxx}} + \dots$$

satisfy

$$\mathrm{D}_x \, a_{ij} = \mathrm{D}_j \, h_i + v_{t_j} \, \mathrm{D}_x \, g_i,$$

where g_i are the right hand sides of the PKdV equations and $h_i = \frac{1}{4i+2} g_{i+1}$.

(b) *There are polynomials b_{ij} in $v, v_x, v_{xx}, \dots$ such that*

$$\mathrm{D}_x \, b_{ij} = g_j \, \mathrm{D}_x \, g_i.$$

Proof. (a) We have

$$\begin{aligned}
\mathrm{D}_x \, a_{ij} &= \mathrm{D}_x \left(v_{t_j} \frac{\delta_1 h_i}{\delta v_x} + v_{xt_j} \frac{\delta_1 h_i}{\delta v_{xx}} + v_{xxt_j} \frac{\delta_1 h_i}{\delta v_{xxx}} + \dots \right) \\
&= v_{xt_j} \frac{\delta_1 h_i}{\delta v_x} + v_{xxt_j} \frac{\delta_1 h_i}{\delta v_{xx}} + v_{xxxt_j} \frac{\delta_1 h_i}{\delta v_{xxx}} + \dots \\
&\quad + v_{t_j} \, \mathrm{D}_x \left(\frac{\delta_1 h_i}{\delta v_x} \right) + v_{xt_j} \, \mathrm{D}_x \left(\frac{\delta_1 h_i}{\delta v_{xx}} \right) + v_{xxt_j} \, \mathrm{D}_x \left(\frac{\delta_1 h_i}{\delta v_{xxx}} \right) + \dots .
\end{aligned}$$

By virtue of Corollary 4.12 it follows that

$$\begin{aligned}
\mathrm{D}_x a_{ij} &= v_{xt_j} \frac{\delta_1 h_i}{\delta v_x} + v_{xxt_j} \frac{\delta_1 h_i}{\delta v_{xx}} + v_{xxxt_j} \frac{\delta_1 h_i}{\delta v_{xxx}} + \dots \\
&\quad - v_{t_j} \frac{\delta_1 h_i}{\delta v} + v_{t_j} \frac{\partial h_i}{\partial v} - v_{xt_j} \frac{\delta_1 h_i}{\delta v_x} + v_{xt_j} \frac{\partial h_i}{\partial v_x} - v_{xxt_j} \frac{\delta_1 h_i}{\delta v_{xx}} + v_{xxt_j} \frac{\partial h_i}{\partial v_{xx}} - \dots \\
&= -v_{t_j} \frac{\delta_1 h_i}{\delta v} + v_{t_j} \frac{\partial h_i}{\partial v} + v_{xt_j} \frac{\partial h_i}{\partial v_x} + v_{xxt_j} \frac{\partial h_i}{\partial v_{xx}} + \dots \\
&= \mathrm{D}_j h_i - v_{t_j} \frac{\delta_1 h_i}{\delta v}
\end{aligned}$$

Finally, by Corollary 4.10 it follows that

$$\mathrm{D}_x a_{ij} = \mathrm{D}_j h_i + v_{t_j} \mathrm{D}_x g_i.$$

(b) We use induction with respect to j. For $j = 0$ we find $g_0 \mathrm{D}_x g_i = \frac{1}{2} \mathrm{D}_x g_i$, so $b_{i0} := \frac{1}{2} g_i$ does the job.

Now assume we have found all b_{ij} for some fixed j. Write the resolvent equation (4.7) for the power series $T_1 := \operatorname{res} \mathcal{T}^+ = \sum_{k=0}^{\infty} r_k z^{-2k-1}$ in terms of its coefficients:

$$\mathrm{D}_x^3 r_k + 4u \mathrm{D}_x r_k + 2u_x r_k = \mathrm{D}_x r_{k+1}.$$

It follows that

$$\mathrm{D}_x^3 g_k + 4v_x \mathrm{D}_x g_k + 2v_{xx} g_k = \mathrm{D}_x g_{k+1}.$$

Now

$$\begin{aligned}
g_{j+1} \mathrm{D}_x g_i &= \mathrm{D}_x (g_{j+1} g_i) - \mathrm{D}_x g_{j+1} \, g_i \\
&= \mathrm{D}_x (g_{j+1} g_i) - \big(\mathrm{D}_x^3 g_j + 4v_x \mathrm{D}_x g_j + 2v_{xx} g_j \big) g_i \\
&= \mathrm{D}_x \Big(g_{j+1} g_i - \mathrm{D}_x^2 g_j \, g_i + \mathrm{D}_x g_j \mathrm{D}_x g_i - g_j \mathrm{D}_x^2 g_i - 4v_x g_j g_i \Big) \\
&\qquad + g_j \big(\mathrm{D}_x^3 g_i + 4v_x \mathrm{D}_x g_i + 2v_{xx} g_i \big) \\
&= g_j \mathrm{D}_x g_{i+1} + \mathrm{D}_x \Big(g_{j+1} g_i - \mathrm{D}_x^2 g_j \, g_i + \mathrm{D}_x g_j \mathrm{D}_x g_i - g_j \mathrm{D}_x^2 g_i - 4v_x g_j g_i \Big),
\end{aligned}$$

so

$$b_{i\,j+1} := b_{i+1\,j} + g_{j+1} g_i - \mathrm{D}_x^2 g_j \, g_i + \mathrm{D}_x g_j \mathrm{D}_x g_i - g_j \mathrm{D}_x^2 g_i - 4v_x g_j g_i$$

does the job. □

Observe that a_{ij} and b_{ij} do not depend on v, only on its derivatives.

Lemma 5.2. *The identity $b_{ij} + b_{ji} = g_i g_j$ holds for all i, j.*

Proof. We know that

$$\mathrm{D}_x(b_{ij} + b_{ji}) = g_j \, \mathrm{D}_x \, g_i + g_i \, \mathrm{D}_x \, g_j = \mathrm{D}_x(g_i g_j).$$

Since neither $b_{ij} + b_{ji}$ nor $g_i g_j$ contain any constant terms, the claim follows. □

5.2. Construction of the Lagrangian two-form

Now we are in a position to construct a pluri-Lagrangian structure for the PKdV hierarchy. We restrict to a finite number of equations, so that our multi-time $\mathbb{R}^N$ remains finite-dimensional. As justified by the first KdV equation, $u_{t_1} = u_x$, we identify t_1 with x.

We look for a Lagrangian two-form

$$\mathcal{L} = \sum_{i<j} L_{ij} \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

where the coefficients

$$L_{1i} := L_i = \frac{1}{2} v_x v_{t_i} - h_i \tag{5.1}$$

are the Lagrangians found before. To determine the other coefficients L_{ij} we will use the fact that the flows of the different PKdV equations are variational symmetries of each other. As in the example of the sine-Gordon equation in Chapter 3, this will give us a closedness property for $\mathcal{L}$.

Fix distinct integers $i, j \in \{2, 3, \ldots, N\}$. Restrict to the (x, t_i)-plane and identify $v_{t_j} = g_j$. We have

$$L_{1i} = \frac{1}{2} v_x v_{t_i} - h_i \qquad \text{and} \qquad L_{1j} = \frac{1}{2} v_x g_j - h_j.$$

Consider the generalized vector field

$$\mathrm{D}_{g_j} := \sum_I (\mathrm{D}_I \, g_j) \frac{\partial}{\partial v_I}$$

and the polynomial

$$a_{ij}^{(g_j)} := g_j \frac{\delta_1 h_i}{\delta v_x} + \mathrm{D}_x \, g_j \frac{\delta_1 h_i}{\delta v_{xx}} + \mathrm{D}_x^2 \, g_j \frac{\delta_1 h_i}{\delta v_{xxx}} + \ldots$$

obtained by identification of v_{t_j} with g_j from D_j and a_{ij} respectively.

Ansatz 5.3. *We want* D_{g_j} *to be a variational symmetry of* L_{1i}*. In particular, we look for* L_{ij} *such that*

$$\mathrm{D}_x L_{ij} - \mathrm{D}_i L_{1j} + \mathrm{D}_{g_j} L_{1i} = 0.$$

We have

$$\mathrm{D}_i L_{1j} = \frac{1}{2} v_{t_i x} g_j + \frac{1}{2} v_x \mathrm{D}_i g_j - \mathrm{D}_i h_j,$$
$$\mathrm{D}_{g_j} L_{1i} = \frac{1}{2} \mathrm{D}_x g_j \, v_{t_i} + \frac{1}{2} v_x \mathrm{D}_i g_j - \mathrm{D}_{g_j} h_i,$$

so

$$\begin{aligned} \mathrm{D}_i L_{1j} - \mathrm{D}_{g_j} L_{1i} &= \frac{1}{2} v_{t_i x} g_j - \frac{1}{2} v_{t_i} \mathrm{D}_x g_j - \mathrm{D}_i h_j + \mathrm{D}_{g_j} h_i \\ &= \frac{1}{2} v_{t_i x} g_j - \frac{1}{2} v_{t_i} \mathrm{D}_x g_j - \mathrm{D}_x a_{ji} + v_{t_i} \mathrm{D}_x g_j + \mathrm{D}_x a_{ij}^{(g_j)} - g_j \mathrm{D}_x g_i \\ &= \frac{1}{2} v_{t_i x} g_j + \frac{1}{2} v_{t_i} \mathrm{D}_x g_j - \mathrm{D}_x \left(a_{ji} - a_{ij}^{(g_j)} \right) - g_j \mathrm{D}_x g_i \\ &= \frac{1}{2} \mathrm{D}_x (v_{t_i} g_j) + \mathrm{D}_x \left(a_{ij}^{(g_j)} - a_{ji} \right) - \mathrm{D}_x b_{ij}. \end{aligned}$$

We denote the antiderivative with respect to x of this quantity by

$$L_{ij}^{(i)} := \frac{1}{2} v_{t_i} g_j + \left(a_{ij}^{(g_j)} - a_{ji} \right) - b_{ij}. \tag{5.2}$$

The analogous calculation in the (x, t_j)-plane yields

$$\mathrm{D}_{g_i} L_{1j} - \mathrm{D}_j L_{1i} = -\frac{1}{2} \mathrm{D}_x (v_{t_j} g_i) + \mathrm{D}_x \left(a_{ij} - a_{ji}^{(g_i)} \right) + \mathrm{D}_x b_{ji}.$$

We denote its antiderivative by

$$L_{ij}^{(j)} := -\frac{1}{2} v_{t_j} g_i + \left(a_{ij} - a_{ji}^{(g_i)} \right) + b_{ji}. \tag{5.3}$$

We look for a function L_{ij} on the three-dimensional (x, t_i, t_j)-space that reduces to $L_{ij}^{(i)}$ and $L_{ij}^{(j)}$ after the substitution $v_{t_j} = g_j$ and $v_{t_i} = g_i$ respectively.

Ansatz 5.4. *The function* L_{ij} *is of the form*

$$L_{ij} = c v_{t_i} v_{t_j} + \sum_{\alpha=0}^{\infty} v_{x^\alpha t_i} \, p_\alpha + \sum_{\alpha=0}^{\infty} v_{x^\alpha t_j} \, q_\alpha + r, \tag{5.4}$$

where $c \in \mathbb{R}$ *is a constant and* p_α*,* q_α*,* r *are differential polynomials in* $v, v_x, v_{xx}, \ldots$*.*

Theorem 5.5. *For every choice of $c \in \mathbb{R}$ there is exactly one such L_{ij}. It is given by*

$$L_{ij} = cv_{t_i}v_{t_j} + (a_{ij} - a_{ji}) + \left(\tfrac{1}{2} - c\right) v_{t_i} g_j - \left(\tfrac{1}{2} + c\right) v_{t_j} g_i - \tfrac{1}{2}(b_{ij} - b_{ji}) + cg_i g_j.$$

Proof. After the substitution $v_{t_j} = g_j$, Equation (5.4) becomes

$$L_{ij}^{(i)} = cv_{t_i} g_j + \sum_{\alpha=0}^{\infty} v_{x^\alpha t_i}\, p_\alpha + \sum_{\alpha=0}^{\infty} q_\alpha \,\mathrm{D}_x^\alpha\, g_j + r,$$

Comparing with Equation (5.2), we find that $p_0 = -\frac{\delta_1 h_j}{\delta v_x} + \left(\frac{1}{2} - c\right) g_j$ and $p_\alpha = -\frac{\delta_1 h_j}{\delta v_{x^{\alpha+1}}}$ for $\alpha > 0$, i.e.

$$\sum_{\alpha=0}^{\infty} v_{x^\alpha t_i}\, p_\alpha = -a_{ji} + (\tfrac{1}{2} - c) v_{t_i} g_j.$$

By substituting $v_{t_i} = g_i$ instead in Equation (5.4), and comparing to Equation (5.3), we find that $q_0 = \frac{\delta_1 h_i}{\delta v_x} - \left(\frac{1}{2} + c\right) g_i$ and $q_\alpha = \frac{\delta_1 h_i}{\delta v_{x^{\alpha+1}}}$ for $\alpha > 0$, i.e.

$$\sum_{\alpha=0}^{\infty} v_{x^\alpha t_j}\, q_\alpha = a_{ij} - \left(\tfrac{1}{2} + c\right) v_{t_j} g_i.$$

This leaves us with

$$L_{ij} = cv_{t_i}v_{t_j} - a_{ji} + a_{ij} + \left(\tfrac{1}{2} - c\right) v_{t_i} g_j - \left(\tfrac{1}{2} + c\right) v_{t_j} g_i + r. \tag{5.5}$$

Now we again do the substitutions and compare with Equations (5.2) and (5.3). We find that

$$r = -b_{ij} + \left(\tfrac{1}{2} + c\right) g_i g_j = b_{ji} - \left(\tfrac{1}{2} - c\right) g_i g_j \tag{5.6}$$

modulo terms that vanish after the substitution $v_{t_i} = g_i$ or $v_{t_j} = g_j$. However, there are no such (nonzero) terms that only depend on $v, v_x, v_{xx}, \ldots$, so Equation (5.6) holds exactly.

The two expressions for r in Equation (5.6) are equal by virtue of Lemma 5.2. In fact, using the same Lemma, we find a more aesthetically pleasing way to express r:

$$r = \tfrac{1}{2}(b_{ji} - b_{ij}) + cg_i g_j.$$

Plugging this into Equation (5.5) yields the desired result. □

Our theory does not depend in any essential way on the choice of L_{ij} among this family. For simplicity we choose $c = 0$, i.e.

$$L_{ij} = \tfrac{1}{2}(v_{t_i} g_j - v_{t_j} g_i) + (a_{ij} - a_{ji}) - \tfrac{1}{2}(b_{ij} - b_{ji}). \tag{5.7}$$

Another interesting choice is to formally put $c = \infty$, or, in other words, to take only the c-linear part. This gives us

$$\widetilde{L}_{1j} = 0 \qquad \text{and} \qquad \widetilde{L}_{ij} = (v_{t_i} - g_i)(v_{t_j} - g_j). \tag{5.8}$$

We will call the form with coefficients (5.1) and (5.7) the *first form* and the one with coefficients (5.8) the *second form*. The second form can be considered for any family of evolution equations $v_{t_j} = g_j$. However, it does not have any connection to the classical Lagrangian formulation of the individual differentiated equations $v_{xt_j} = \mathrm{D}_x\, g_j$.

The reason we assumed Ansatz 5.3 is that it gives us the following closedness property.

Proposition 5.6. (a) *The two-form $\mathcal{L} = \sum_{i<j} L_{ij}\, \mathrm{d}t_i \wedge \mathrm{d}t_j$, with coefficients given by* (5.1) *and* (5.7)*, is closed as soon as v solves all but one of the PKdV equations.*

(b) *The same property holds when the coefficients are given by* (5.8)*.*

Proof. Consider the first form, i.e. the one with coefficients given by (5.1) and (5.7). The coefficients of $\mathrm{d}\mathcal{L}$ are given by $\mathrm{D}_k\, L_{ij} - \mathrm{D}_j\, L_{ik} + \mathrm{D}_i\, L_{jk}$ for all $i < j < k$.

Consider the case that $i = 1$. By construction (Ansatz 5.3), $\mathrm{D}_k\, L_{1j} - \mathrm{D}_j\, L_{1k} + \mathrm{D}_x\, L_{jk}$ vanishes as soon as either $v_{t_j} = g_j$ or $v_{t_k} = g_k$ is satisfied. Indeed, we have that

$$\begin{aligned}
&\mathrm{D}_k L_{1j} - \mathrm{D}_j\, L_{1k} + \mathrm{D}_x\, L_{jk} \\
&\quad = \frac{1}{2} v_{t_j t_k} v_x + \frac{1}{2} v_{t_j} v_{x t_k} - \mathrm{D}_k\, h_j - \frac{1}{2} v_{t_j t_k} v_x - \frac{1}{2} v_{t_k} v_{x t_j} + \mathrm{D}_j\, h_k \\
&\qquad + \frac{1}{2}\big(v_{x t_j} g_k + v_{t_j}\, \mathrm{D}_x\, g_k - v_{x t_k} g_j - v_{t_k}\, \mathrm{D}_x\, g_j \big) \\
&\qquad + \mathrm{D}_k\, h_j + v_{t_k}\, \mathrm{D}_x\, g_j - \mathrm{D}_j\, h_k - v_{t_j}\, \mathrm{D}_x\, g_k - \frac{1}{2}(g_k\, \mathrm{D}_x\, g_j - g_j\, \mathrm{D}_x\, g_k) \\
&\quad = \frac{1}{2}\big(v_{t_j} v_{x t_k} - v_{t_k} v_{x t_j} + v_{x t_j} g_k - v_{t_j}\, \mathrm{D}_x\, g_k - v_{x t_k} g_j + v_{t_k}\, \mathrm{D}_x\, g_j - g_k\, \mathrm{D}_x\, g_j + g_j\, \mathrm{D}_x\, g_k \big) \\
&\quad = \frac{1}{2}(v_{t_j} - g_j)\, \mathrm{D}_x (v_{t_k} - g_k) - \frac{1}{2}(v_{t_k} - g_k)\, \mathrm{D}_x (v_{t_j} - g_j).
\end{aligned} \tag{5.9}$$

If $i, j, k > 1$ we can assume without loss of generality that $v_{t_i} = g_i$ and $v_{t_j} = g_j$ are satisfied. Regardless of whether the corresponding equation for v_{t_k} holds, we do not make any identification involving $v_{t_k}, v_{x t_k}, \ldots$. Using Equation (5.9) we find

$$\begin{aligned}
\mathrm{D}_x \left(\mathrm{D}_k\, L_{ij} - \mathrm{D}_j\, L_{ik} + \mathrm{D}_i\, L_{jk} \right) = {} & \mathrm{D}_k \left(\mathrm{D}_i\, L_{1j} - \mathrm{D}_j\, L_{1i} \right) \\
& - \mathrm{D}_j \left(\mathrm{D}_i\, L_{1k} - \mathrm{D}_k\, L_{1i} \right) \\
& + \mathrm{D}_i \left(\mathrm{D}_j\, L_{1k} - \mathrm{D}_k\, L_{1j} \right) = 0.
\end{aligned}$$

Since these polynomials do not contain constant terms, it follows that

$$\mathrm{D}_k\, L_{ij} - \mathrm{D}_j\, L_{ik} + \mathrm{D}_i\, L_{jk} = 0.$$

For the second form, where the coefficients are given by Equation (5.8), the claim is trivial. □

5.3. The multi-time Euler-Lagrange equations

Theorem 5.7. (a) *The multi-time Euler-Lagrange equations for the first Lagrangian two-form* $\mathcal{L} = \sum_{i<j} L_{ij} \, \mathrm{d}t_i \wedge \mathrm{d}t_j$*, with coefficients given by* (5.1) *and* (5.7)*, are the first* $N-1$ *nontrivial PKdV equations*

$$v_{t_2} = g_2, \quad v_{t_3} = g_3, \quad \dots \quad v_{t_N} = g_N,$$

and equations that follow from these.

(b) *The same result holds for the second form given by* (5.8)*.*

The proof of Theorem 5.7 consists of checking all multi-time Euler-Lagrange equations (2.5)–(2.7) individually for both forms, and will take up the rest of this section. If $N > 3$ we fix $k > j > i > 1$. If $N = 3$ we take $j = 3$, $i = 2$ and in the following ignore all equations containing k. We use the convention $L_{ji} = -L_{ij}$, etc.

First form, coefficients (5.1) and (5.7)

Equations (2.7)

- The equations

$$\frac{\delta_{1i} L_{1i}}{\delta v_{Ixt_i}} + \frac{\delta_{ij} L_{ij}}{\delta v_{It_it_j}} + \frac{\delta_{1j} L_{j1}}{\delta v_{It_jx}} = 0$$

  and

$$\frac{\delta_{ij} L_{ij}}{\delta v_{It_it_j}} + \frac{\delta_{jk} L_{jk}}{\delta v_{It_jt_k}} + \frac{\delta_{ki} L_{ki}}{\delta v_{It_kt_i}} = 0$$

  are trivial because all terms vanish.

Equations (2.6)

- The equation

$$\frac{\delta_{1i} L_{1i}}{\delta v_x} = \frac{\delta_{ij} L_{ji}}{\delta v_{t_j}}$$

yields

$$\begin{aligned}\frac{1}{2}v_{t_i} - \frac{\delta_{1i} h_i}{\delta v_x} &= \frac{1}{2}g_i - \frac{\delta_{ij} a_{ij}}{\delta v_{t_j}} \\ &= \frac{1}{2}g_i - \frac{\delta_{ij}}{\delta v_{t_j}} \left(v_{t_j} \frac{\delta_1 h_i}{\delta v_x} + v_{t_j x} \frac{\delta_1 h_i}{\delta v_{xx}} + v_{t_j xx} \frac{\delta_1 h_i}{\delta v_{xxx}} + \dots \right) \\ &= \frac{1}{2}g_i - \frac{\delta_1 h_i}{\delta v_x}.\end{aligned}$$

This simplifies to the PKdV equation

$$v_{t_i} = g_i. \tag{5.10}$$

- For $\alpha > 0$, the equation

$$\frac{\delta_{1i} L_{1i}}{\delta v_{x^{\alpha+1}}} = \frac{\delta_{ij} L_{ji}}{\delta v_{t_j x^\alpha}}$$

  yields

$$\begin{aligned}-\frac{\delta_{1i} h_i}{\delta v_{x^{\alpha+1}}} &= -\frac{\delta_{ij}}{\delta v_{t_j x^\alpha}} \left(v_{t_j} \frac{\delta_1 h_i}{\delta v_x} + v_{t_j x} \frac{\delta_1 h_i}{\delta v_{xx}} + v_{t_j xx} \frac{\delta_1 h_i}{\delta v_{xxx}} + \dots \right) \\ &= -\frac{\delta_1 h_i}{\delta v_{x^{\alpha+1}}},\end{aligned}$$

  which is trivial.

- Similarly, the equation

$$\frac{\delta_{1j} L_{1j}}{\delta v_x} = \frac{\delta_{ij} L_{ij}}{\delta v_{t_i}}$$

  yields PKdV equation

$$v_{t_j} = g_j, \tag{5.11}$$

  and for $\alpha > 0$, the equation

$$\frac{\delta_{1j} L_{1j}}{\delta v_{x^{\alpha+1}}} = \frac{\delta_{ij} L_{ij}}{\delta v_{t_i x^\alpha}}$$

  is trivial.

- All equations of the form

$$\frac{\delta_{1i} L_{1i}}{\delta v_{xI}} = \frac{\delta_{ij} L_{ji}}{\delta v_{t_j I}} \quad (t_i \notin I) \qquad \text{and} \qquad \frac{\delta_{1j} L_{1j}}{\delta v_{xI}} = \frac{\delta_{ij} L_{ij}}{\delta v_{t_i I}} \quad (t_j \notin I)$$

  where I contains any t_l $(l > 1)$ are trivial because each term is zero.

- The equations
$$\frac{\delta_{1i} L_{1i}}{\delta v_{It_i}} = \frac{\delta_{1j} L_{1j}}{\delta v_{It_j}} \qquad (x \notin I)$$
are trivial because both sides are zero for nonempty I and both are equal to $\frac{1}{2} v_x$ for empty I.

Equations (2.5)

- By construction, the equations $\dfrac{\delta_{1i} L_{1i}}{\delta v} = 0$ are the DPKdV equations
$$v_{xt_i} = \mathrm{D}_x\, g_i. \tag{5.12}$$
For I containing any t_l, $l > 1$, $l \neq i$, the equations $\dfrac{\delta_{1i} L_{1i}}{\delta v_I} = 0$ are trivial.

- We discuss the last family of equations as a lemma because its calculation is far from trivial.
Lemma 5.8. *The equations* $\dfrac{\delta_{ij} L_{ij}}{\delta v_{x^\alpha}} = 0$ *are corollaries of the PKdV equations.*

At first sight there is nothing in our construction that forces this claim to be true. So before proceeding with the proof, we present a heuristic argument why the claim should hold.

Consider a small three-dimensional cube C in $\mathbb{R}^N$, and a solution v to the PKdV equations. As we have already calculated, v satisfies all multi-time Euler-Lagrange equations except possibly $\frac{\delta_{ij} L_{ij}}{\delta v_{x^\alpha}} = 0$ for all α. Hence for any variation $\mathscr{V}$, the variation of the action (Equation (2.12)) on the boundary of the cube simplifies to
$$\int_{\partial C} \iota_{\mathrm{pr}\, \mathscr{V}}\, \delta \mathcal{L} = \int_{\partial C} \sum_{\alpha \in \mathbb{N}} \frac{\delta_{ij} L_{ij}}{\delta v_{x^\alpha}} \delta v_{x^\alpha}(\mathscr{V})\, \mathrm{d}t_i \wedge \mathrm{d}t_j.$$

If we choose the variations along simultaneous solutions of all but one of the PKdV equations, $\mathcal{L}$ will be closed along these variations, i.e. $\iota_{\mathrm{pr}\, \mathscr{V}}\, \delta \mathcal{L}$ will be closed. Therefore
$$\int_{\partial C} \sum_{\alpha \in \mathbb{N}} \frac{\delta_{ij} L_{ij}}{\delta v_{x^\alpha}} \delta v_{x^\alpha}(\mathscr{V})\, \mathrm{d}t_i \wedge \mathrm{d}t_j = \int_{\partial C} \iota_{\mathrm{pr}\, \mathscr{V}}\, \delta \mathcal{L} = \int_C \mathrm{d}(\iota_{\mathrm{pr}\, \mathscr{V}}\, \delta \mathcal{L}) = 0,$$
so the equations $\dfrac{\delta_{ij} L_{ij}}{\delta v_{x^\alpha}} = 0$ for $k \geq 0$ are satisfied.

The reason this argument is only a heuristic one, is that the set of variations that we consider might be too small.

Proof of Lemma 5.8. From Equation (5.7) we see that the variational derivative of L_{ij} contains only three nonzero terms,

$$\frac{\delta_{ij} L_{ij}}{\delta v_{x^\alpha}} = \frac{\partial L_{ij}}{\partial v_{x^\alpha}} - \mathrm{D}_i \left(\frac{\partial L_{ij}}{\partial v_{x^\alpha t_i}} \right) - \mathrm{D}\, j \left(\frac{\partial L_{ij}}{\partial v_{x^\alpha t_j}} \right). \tag{5.13}$$

To determine the first term we use an indirect method. Assume that the dimension of multi-time N is at least 4 and fix $k > 1$ distinct from i and j. Let v be a solution of all PKdV equations except $v_{t_k} = g_k$. By Proposition 5.6 we have

$$\sum_I \frac{\partial L_{ij}}{\partial v_I} v_{I t_k} = \mathrm{D}_k\, L_{ij} = \mathrm{D}_j\, L_{ik} - \mathrm{D}_i\, L_{jk}. \tag{5.14}$$

Since $\frac{\partial L_{ij}}{\partial v_I}$ does not contain any derivatives with respect to t_k, we can determine $\frac{\partial L_{ij}}{\partial v_{x^\alpha}}$ by looking at the terms in the right hand side of Equation (5.14) containing $v_{x^\alpha t_k}$. These are

$$\begin{aligned} &\mathrm{D}_j \left(-\tfrac{1}{2} g_i v_{t_k} + v_{t_k} \frac{\delta_1 h_i}{\delta v_x} + v_{x t_k} \frac{\delta_1 h_i}{\delta v_{xx}} + \dots \right) \\ - &\mathrm{D}_i \left(-\tfrac{1}{2} g_j v_{t_k} + v_{t_k} \frac{\delta_1 h_j}{\delta v_x} + v_{x t_k} \frac{\delta_1 h_j}{\delta v_{xx}} + \dots \right). \end{aligned}$$

Now we expand the brackets. By again throwing out all terms that do not contain any $v_{x^\alpha t_k}$, and those that cancel modulo $v_{t_i} = g_i$ or $v_{t_j} = g_j$, we get

$$\begin{aligned} &+ v_{t_k}\, \mathrm{D}_j \left(\frac{\delta_1 h_i}{\delta v_x} \right) + v_{x t_k}\, \mathrm{D}_j \left(\frac{\delta_1 h_i}{\delta v_{xx}} \right) + v_{xx t_k}\, \mathrm{D}_j \left(\frac{\delta_1 h_i}{\delta v_{xxx}} \right) + \dots \\ &- v_{t_k}\, \mathrm{D}_i \left(\frac{\delta_1 h_j}{\delta v_x} \right) - v_{x t_k}\, \mathrm{D}_i \left(\frac{\delta_1 h_j}{\delta v_{xx}} \right) - v_{xx t_k}\, \mathrm{D}_i \left(\frac{\delta_1 h_j}{\delta v_{xxx}} \right) - \dots. \end{aligned}$$

Comparing this to Equation (5.14), we find that

$$\frac{\partial L_{ij}}{\partial v_{x^\alpha}} = - \mathrm{D}_i \left(\frac{\delta_1 h_j}{\delta v_{x^{\alpha+1}}} \right) + \mathrm{D}_j \left(\frac{\delta_1 h_i}{\delta v_{x^{\alpha+1}}} \right).$$

On the other hand we have

$$- \mathrm{D}_i \left(\frac{\partial L_{ij}}{\partial v_{x^\alpha t_i}} \right) - \mathrm{D}_j \left(\frac{\partial L_{ij}}{\partial v_{x^\alpha t_j}} \right) = \mathrm{D}_i \left(\frac{\delta_1 h_j}{\delta v_{x^{\alpha+1}}} \right) - \mathrm{D}_j \left(\frac{\delta_1 h_i}{\delta v_{x^{\alpha+1}}} \right),$$

so Equation (5.13) implies that $\dfrac{\delta_{ij} L_{ij}}{\delta v_{x^\alpha}} = 0$.

Since $\frac{\delta_{23} L_{23}}{\delta v_{x^\alpha}} = 0$ does not depend on the dimension $N \geq 3$, the result for $N \geq 4$ implies the claim for $N = 3$. □

Second form, coefficients (5.8)

Equations (2.7)

- The equations

$$\frac{\delta_{1i} \widetilde{L}_{1i}}{\delta v_{Ixt_i}} + \frac{\delta_{ij} \widetilde{L}_{ij}}{\delta v_{It_it_j}} + \frac{\delta_{1j} \widetilde{L}_{j1}}{\delta v_{It_jx}} = 0$$

  and

$$\frac{\delta_{ij} \widetilde{L}_{ij}}{\delta v_{It_it_j}} + \frac{\delta_{jk} \widetilde{L}_{jk}}{\delta v_{It_jt_k}} + \frac{\delta_{ki} \widetilde{L}_{ki}}{\delta v_{It_kt_i}} = 0$$

  are trivial because all terms vanish.

Equations (2.6)

- The equations

$$\frac{\delta_{1i} \widetilde{L}_{1i}}{\delta v_x} = \frac{\delta_{ij} \widetilde{L}_{ji}}{\delta v_{t_j}} \qquad \text{and} \qquad \frac{\delta_{1j} \widetilde{L}_{1j}}{\delta v_x} = \frac{\delta_{ij} \widetilde{L}_{ij}}{\delta v_{t_i}}$$

  yield the PKdV equations

$$0 = g_i - v_{t_i} \qquad \text{and} \qquad 0 = v_{t_j} - g_j \tag{5.15}$$

  respectively.

- All equations of the form

$$\frac{\delta_{1i} \widetilde{L}_{1i}}{\delta v_{xI}} = \frac{\delta_{ij} \widetilde{L}_{ji}}{\delta v_{t_jI}} \quad (t_i \notin I) \qquad \text{and} \qquad \frac{\delta_{1j} \widetilde{L}_{1j}}{\delta v_{xI}} = \frac{\delta_{ij} \widetilde{L}_{ij}}{\delta v_{t_iI}} \quad (t_j \notin I)$$

  where I is nonempty are trivial because each term is zero.

- The equations $\frac{\delta_{1i} \widetilde{L}_{1i}}{\delta v_{It_i}} = \frac{\delta_{1j} \widetilde{L}_{1j}}{\delta v_{It_j}}$, with $x \notin I$, are trivial because both sides are zero.

Equations (2.5)

- The equations $\frac{\delta_{1i} \widetilde{L}_{1i}}{\delta v_I} = 0$, with $x \notin I$ and $t_i \notin I$, are trivial for every such I.

- The equations

$$\frac{\delta_{ij}\widetilde{L}_{ij}}{\delta v_{x^\alpha}} = 0$$

yield

$$-\frac{\partial g_i}{\partial v_{x^\alpha}}(v_{t_j} - g_j) + \frac{\partial g_j}{\partial v_{x^\alpha}}(v_{t_i} - g_i) = 0,$$

which are corollaries of the PKdV equations (5.15).

For I containing any t_l with $l > 1$ and $l \neq i$, the equations $\dfrac{\delta_{ij}\widetilde{L}_{ij}}{\delta v_I} = 0$ are trivial.

This concludes the proof of Theorem 5.7.

It is remarkable that multi-time Euler-Lagrange equations are capable of producing first order evolution equations. This is a striking difference from both the traditional variational theories and the discrete case of the pluri-Lagrangian theory. In the traditional Lagrangian formalism, the variational derivative does not produce terms of the form v_t (see Appendix A.2), hence the Euler-Lagrange equations cannot be first order evolution equations. In pluri-Lagrangian lattice systems the evolution equations (*quad equations*) imply the multi-time Euler-Lagrange equations (*corner equations*), but they themselves are not variational [5].

6. Relation to Hamiltonian formalism

In Proposition 2.2 we saw that $\mathrm{d}\mathcal{L}$ is constant on solutions. For the one-dimensional case ($d = 1$) with $\mathcal{L}$ depending on the first jet bundle only, Suris [15] has shown that this is equivalent to the commutativity of the corresponding Hamiltonian flows. Moreover, if the constant is zero, then the Hamiltonians are in involution. Now we will prove a similar result for the two-dimensional case.

We will use a Poisson bracket on *formal integrals*, i.e. equivalence classes of functions modulo x-derivatives [6, Chapter 1–2]. It is defined by

$$\left\{ \textstyle\int F, \int G \right\} := \int \left(\mathrm{D}_x \frac{\delta_1 F}{\delta u} \right) \frac{\delta_1 G}{\delta u}.$$

Using integration by parts we see that this bracket is anti-symmetric. Less obvious is the fact that it satisfies the Jacobi identity [13, Chapter 7]. In this section, the integral sign $\int$ will always denote an equivalence class, not an integration operator.

As we did when studying the KdV hierarchy, we introduce a potential v that satisfies $v_x = u$ and we identify the space-coordinate x with the first coordinate t_1 of multi-time. We can now re-write the Poisson bracket as

$$\left\{ \textstyle\int F, \int G \right\} = \int \left(\mathrm{D}_x \frac{\delta_1 F}{\delta v_x} \right) \frac{\delta_1 G}{\delta v_x} = - \int \frac{\delta_1 F}{\delta v} \frac{\delta_1 G}{\delta v_x}, \tag{6.1}$$

for functions F and G that depend on the x-derivatives of v but not on v itself.

Assume that the coefficients L_{1j} of the Lagrangian two-from $\mathcal{L}$ have the form

$$L_{1j} = \frac{1}{2} v_x v_{t_j} - h_j,$$

where h_j is a differential polynomial in $v_x, v_{xx}, \ldots$. This is consistent with the first Lagrangian two-form of the PKdV hierarchy. The L_{1j} are Lagrangians of the equations

$$v_{x t_j} = \mathrm{D}_x g_j \qquad \text{or} \qquad u_{t_j} = \mathrm{D}_x g_j,$$

where $g_j := \frac{\delta_1 h_j}{\delta v_x}$, hence $\frac{\delta_1 h_j}{\delta v} = - \mathrm{D}_x g_j$. It turns out that the formal integral $\int h_j$ is the Hamilton functional for the equation $u_{t_j} = \mathrm{D}_x g_j$ with respect to the Poisson bracket

(6.1). Formally:

$$\left\{\textstyle\int h_j, u(y)\right\} = \left\{\textstyle\int h_j, \int v_x\, \delta(\cdot - y)\right\} = -\int \frac{\delta_1 h_j}{\delta v} \delta(\cdot - y) = \mathrm{D}_x\, g_j(y),$$

where δ denotes the Dirac delta.

Theorem 6.1. *If* $\mathrm{d}\mathcal{L} = 0$ *on solutions of the evolution equations* $v_{t_j} = g_j$*, then the Hamiltonians are in involution,*

$$\left\{\textstyle\int h_i, \int h_j\right\} = 0.$$

Proof. Write $\mathrm{d}\mathcal{L} = \sum_{i<j<k} M_{ijk}\, \mathrm{d}t_i \wedge \mathrm{d}t_j \wedge \mathrm{d}t_k$. We have

$$\begin{aligned}
\int M_{1jk}\, \mathrm{d}x &= \int \big(\mathrm{D}_x\, L_{jk} - \mathrm{D}_j\, L_{1k} + \mathrm{D}_k\, L_{1j} \big) \\
&= \int \big(- \mathrm{D}_j\, L_{1k} + \mathrm{D}_k\, L_{1j} \big) \\
&= \int \left(-\frac{1}{2} v_{xt_j} v_{t_k} - \frac{1}{2} v_x v_{t_k t_j} + \mathrm{D}_j\, h_k + \frac{1}{2} v_{xt_k} v_{t_j} + \frac{1}{2} v_x v_{t_j t_k} - \mathrm{D}_k\, h_j \right) \\
&= \int \left(\frac{1}{2} (v_{xt_k} v_{t_j} - v_{xt_j} v_{t_k}) - \mathrm{D}_k\, h_j + \mathrm{D}_j\, h_k \right).
\end{aligned}$$

Using Lemma 5.1(a) (which, as opposed to Lemma 5.1(b), is independent of the form of h_i and g_i), the evolution equations $v_{t_j} = g_j$, and integration by parts, we find that

$$\begin{aligned}
\int M_{1jk}\, \mathrm{d}x &= \int \left(\frac{1}{2} (v_{xt_k} v_{t_j} - v_{xt_j} v_{t_k}) - \mathrm{D}_x\, a_{jk} + v_{t_k}\, \mathrm{D}_x\, g_j + \mathrm{D}_x\, a_{kj} - v_{t_j} D_x g_k \right) \\
&= \int \left(-\frac{1}{2} (g_j\, \mathrm{D}_x\, g_k - g_k\, \mathrm{D}_x\, g_j) - \mathrm{D}_x\, a_{jk} + \mathrm{D}_x\, a_{kj} \right) \\
&= \int g_k\, \mathrm{D}_x\, g_j \\
&= -\int \frac{\delta_1 h_j}{\delta v} \frac{\delta_1 h_k}{\delta v_x} \\
&= \left\{ \textstyle\int h_j, \int h_k \right\}.
\end{aligned}$$

Hence if $\mathrm{d}\mathcal{L} = 0$ on solutions of the evolution equations $v_{t_j} = g_j$, then the Hamilton functionals are in involution. □

7. Conclusion

The truth is rarely pure and never simple

Oscar Wilde

In this thesis I have formulated the pluri-Lagrangian theory of integrable hierarchies, and together with Yuri Suris I propose it as a definition of integrability [17]. The motivation for this definition comes from the discrete case [5, 10, 15] and the fact that there is a relation with the Hamiltonian side of the theory. For the Hamiltonians to be in involution, we need the additional property that the Lagrangian two-form is closed on solutions. However, I am not aware of any examples were this is not the case. Indeed, in all examples considered in this text, the two-form is closed as soon as all but one of the equations are satisfied.

Since the (P)KdV hierarchy is one of the most important examples of an integrable hierarchy, the the fact that it possesses a pluri-Lagrangian structure is an additional indication that the existence of a pluri-Lagrangian structure is a reasonable definition of integrability.

On the other hand, the second Lagrangian two-form we found for the KdV hierarchy is too general to be a characterization of integrability. It can be considered for any family of first order evolution equations. Furthermore, the link to Hamiltonian theory does not apply to this form. Therefore it would make sense to exclude such a form from the definition of a pluri-Lagrangian system. This could be done by introducing a weak extra criterion, e.g. that none of the coefficients of the Lagrangian two-form are identically zero, or that the two-form does not vanish on solutions.

In contrast to the discrete case, continuous pluri-Lagrangian systems are capable of producing evolutionary equations. This discrepancy invites additional research. Another question that is left for future to decide, is whether the pluri-Lagrangian formulation provides any new insight in the theory integrable systems. However, it seems unlikely to me that such a nice theory would remain without applications.

Appendix

A.1. A very short introduction to the variational bicomplex

Here we introduce the variational bicomplex and state the basic results that are used in the text. We follow Dickey, who provides a more complete discussion in [6, Chapter 19]. Another good source on a (subtly different) variational bicomplex is Anderson's unfinished manuscript [1]. For ease of notation we restrict to real fields $u : \mathbb{R}^N \to \mathbb{R}$, rather than vector-valued fields.

The space of (p,q)-forms $\mathcal{A}^{(p,q)}$ consists of all formal sums

$$\omega^{p,q} = \sum f \, \delta u_{I_1} \wedge \ldots \wedge \delta u_{I_p} \wedge \mathrm{d}t_{j_1} \wedge \ldots \wedge \mathrm{d}t_{j_q},$$

where f is a polynomial in $t_1, \ldots, t_N$, v, and partial derivatives of v with respect to any coordinates. We call $(0,q)$-forms *horizontal* and $(p,0)$-forms *vertical*. The vertical one-forms δu_I are dual to the vector fields $\frac{\partial}{\partial u_I}$. The action of the derivative D_i on the (p,q)-form $\omega^{p,q}$ is

$$\begin{aligned} \mathrm{D}_i \, \omega^{p,q} := \sum (\mathrm{D}_i \, f) \, \delta u_{I_1} \wedge \ldots \wedge \delta u_{I_p} \wedge \mathrm{d}t_{j_1} \wedge \ldots \wedge \mathrm{d}t_{j_q} \\ + f \, \delta u_{I_1 i} \wedge \ldots \wedge \delta u_{I_p} \wedge \mathrm{d}t_{j_1} \wedge \ldots \wedge \mathrm{d}t_{j_q} \\ + \ldots \\ + f \, \delta u_{I_1} \wedge \ldots \wedge \delta u_{I_p i} \wedge \mathrm{d}t_{j_1} \wedge \ldots \wedge \mathrm{d}t_{j_q}. \end{aligned}$$

The integral of $\omega^{p,q}$ over an q-dimensional manifold is the $(p,0)$-form defined by

$$\int \omega^{p,q} := \sum \left(\int f \, \mathrm{d}t_{j_1} \wedge \ldots \wedge \mathrm{d}t_{j_q} \right) \delta u_{I_1} \wedge \ldots \wedge \delta u_{I_p}.$$

The *horizontal exterior derivative* $\mathrm{d} : \mathcal{A}^{(p,q)} \to \mathcal{A}^{(p,q+1)}$ and the *vertical exterior derivative* $\delta : \mathcal{A}^{(p,q)} \to \mathcal{A}^{(p+1,q)}$ are defined by the anti-derivation property

(a) $\mathrm{d} \, (\omega_1^{p_1,q_1} \wedge \omega_2^{p_2,q_2}) = \mathrm{d}\omega_1^{p_1,q_1} \wedge \omega_2^{p_2,q_2} + (-1)^{p_1+q_1} \, \omega_1^{p_1,q_1} \wedge \mathrm{d}\omega_2^{p_2,q_2},$

$\delta \, (\omega_1^{p_1,q_1} \wedge \omega_2^{p_2,q_2}) = \delta\omega_1^{p_1,q_1} \wedge \omega_2^{p_2,q_2} + (-1)^{p_1+q_1} \, \omega_1^{p_1,q_1} \wedge \delta\omega_2^{p_2,q_2},$

and by the way they act on $(0,0)$-, $(1,0)$-, and $(0,1)$-forms:

(b) $\mathrm{d}f = \sum_j \mathrm{D}_j \, f \, \mathrm{d}t_j = \sum_j \left(\frac{\partial f}{\partial t_j} + \sum_I \frac{\partial f}{\partial u_I} u_{Ij} \right) \mathrm{d}t_j$ and

$\delta f = \sum_I \frac{\partial f}{\partial u_I} \delta u_I$ for polynomials f in t_j, v, and partial derivatives of v,

(c) $\mathrm{d}\delta u_I = -\sum_j \delta u_{Ij} \wedge \mathrm{d}t_j \qquad \text{and} \qquad \delta(\delta u_I) = 0,$

(d) $\mathrm{d}(\mathrm{d}t_j) = 0 \qquad \text{and} \qquad \delta(\mathrm{d}t_j) = 0.$

Properties (a)–(d) determine the action of d and δ on any form. In particular, we have

$$\delta \mathrm{d}u_I = \delta\left(\sum_j u_{Ij}\,\mathrm{d}t_j\right) = \sum_j \delta u_{Ij} \wedge \mathrm{d}t_j = -\mathrm{d}\delta u_I.$$

The corresponding mapping diagram is known as the *variational bicomplex.*

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & & & \vdots & & \vdots \\
\uparrow\delta & & \uparrow\delta & & & & \uparrow\delta & & \uparrow\delta \\
\mathcal{A}^{(1,0)} & \stackrel{\mathrm{d}}{\to} & \mathcal{A}^{(1,1)} & \stackrel{\mathrm{d}}{\to} & \dots & \stackrel{\mathrm{d}}{\to} & \mathcal{A}^{(1,n-1)} & \stackrel{\mathrm{d}}{\to} & \mathcal{A}^{(1,n)} \\
\uparrow\delta & & \uparrow\delta & & & & \uparrow\delta & & \uparrow\delta \\
\mathcal{A}^{(0,0)} & \stackrel{\mathrm{d}}{\to} & \mathcal{A}^{(0,1)} & \stackrel{\mathrm{d}}{\to} & \dots & \stackrel{\mathrm{d}}{\to} & \mathcal{A}^{(0,n-1)} & \stackrel{\mathrm{d}}{\to} & \mathcal{A}^{(0,n)}
\end{array}$$

The following claims follow by elementary calculations from properties (a)–(d).

Proposition A.1. *We have* $\mathrm{d}^2 = \delta^2 = 0$ *and* $\mathrm{d}\delta + \delta\mathrm{d} = 0$.

Remark. This implies that $\mathrm{d}+\delta : \mathcal{A}^k \to \mathcal{A}^{k+1}$, where $\mathcal{A}^k := \bigcup_{i=0}^k \mathcal{A}^{(i,k-i)}$, is an exterior derivative as well.

Proposition A.2. *We have* $\mathrm{D}_i\,\delta = \delta\,\mathrm{D}_i$

Proposition A.3. *For a differential polynomial* h*, define* $\partial_h := \sum_I (\mathrm{D}_I\, h)\frac{\partial}{\partial u_I}$*. We have* $\mathrm{d}\,\iota_{\partial_h} + \iota_{\partial_h}\,\mathrm{d} = 0$.

Proof. It suffices to show this for every polynomial f in t_j, v, and partial derivatives of v, for $\mathrm{d}t_j$, and for δu_I. For the first two, both terms of the claimed identity are zero,

$$\mathrm{d}(\iota_{\partial_h} f) = 0, \quad \iota_{\partial_h}(\mathrm{d}f) = \iota_{\partial_h}\left(\sum_j \mathrm{D}_j\, f\,\mathrm{d}t_j\right) = 0, \quad \mathrm{d}(\iota_{\partial_h}\,\mathrm{d}t_j) = 0, \quad \iota_{\partial_h}(\mathrm{d}\mathrm{d}t_j) = 0.$$

For vertical one-forms we find

$$\iota_{\partial_h}(\mathrm{d}\delta u_I) = \iota_{\partial_h}\left(-\sum_j \delta u_{Ij} \wedge \mathrm{d}t_j\right) = -\sum_j \mathrm{D}_{Ij}\,h\,\mathrm{d}t_j = -\mathrm{d}(\mathrm{D}_I\,h) = -\mathrm{d}(\iota_{\partial_h}\,\delta u_I). \quad \square$$

A.2. Traditional Euler-Lagrange equations are not first order evolution equations

The traditional Lagrangian formalism does not produce first order evolution equations $u_t = \ldots$. The following proposition explains why.

Proposition A.4. *Let f be a differential polynomial on $\mathbb{R}^N(t_1, \ldots, t_N)$. The variational derivative $\frac{\delta_{1\ldots N} f}{\delta u}$ does not contain terms of the form u_{t_i}.*

Proof. Consider a monomial $m := u_{I_1} \ldots u_{I_n}$. We call n its *degree* and $|I_1| + \ldots + |I_n|$ its *differential order.*

All terms of the variational derivative

$$\frac{\delta_{1\ldots N}\, m}{\delta u} = \sum_J (-1)^{|J|}\, \mathrm{D}_J \left(\frac{\partial m}{\partial u_J} \right)$$

have degree $n-1$ and differential order $|I_1| + \ldots + |I_n|$. Hence, the only way $\frac{\delta f}{\delta u}$ could possibly contain a term u_{t_i} is if f contains a second degree term of first order. Such a term must be of the form uu_{t_i}, but that is a full t_i-derivative, so its variational derivative vanishes,

$$\frac{\delta_{1\ldots N}\, uu_{t_i}}{\delta u} = u_{t_i} - \mathrm{D}_i\, u = 0. \qquad \square$$

As we have seen in Chapters 3 and 5, the pluri-Lagrangian formalism does in fact produce first order evolution equations. It is easy to see where the above proof fails in these cases. If $i \notin I$, then the variational derivative

$$\frac{\delta_I\, uu_{t_i}}{\delta u}$$

does not vanish. Instead it yields exactly u_{t_i}.

A.3. Construction of the KdV hierarchy from the resolvent equation

In this section we mainly follow Dickey [6, Section 3.7]. One way to introduce the Korteweg-de Vries Hierarchy is to consider a formal power series

$$R = \sum_{k=0}^{\infty} r_k z^{-2k-1}$$

satisfying the resolvent equation

$$\mathrm{D}_x^3 R + 4u \,\mathrm{D}_x R + 2u_x R - z^2 \,\mathrm{D}_x R = 0. \tag{A.1}$$

Multiplying this equation by R and integrating with respect to x we find

$$R \,\mathrm{D}_x^2 R - \frac{1}{2}(\mathrm{D}_x R)^2 + 2\left(u - \frac{1}{4}z^2\right) R^2 = C(z), \tag{A.2}$$

where $C(z) = \sum_{k=0}^{\infty} c_k z^{-2k}$ is a formal power series in z^{-2}, with constant real coefficients c_k.

Proposition A.5. *For every C with $c_0 \geq 0$ there exists a real solution to Equation* (A.2).

Proof. Equation (A.2) provides a recurrence relation for r_k. Indeed, we have

$$\begin{aligned}
R \,\mathrm{D}_x^2 R &= \sum_{l=0}^{\infty} \left(\sum_{m=0}^{l} r_{l-m} \,\mathrm{D}_x^2 r_m \right) z^{-2l-2}, \\
\frac{1}{2}(\mathrm{D}_x R)^2 &= \sum_{l=0}^{\infty} \left(\sum_{m=0}^{l} \frac{1}{2} \,\mathrm{D}_x r_{l-m} \,\mathrm{D}_x r_m \right) z^{-2l-2}, \\
2uR^2 &= \sum_{l=0}^{\infty} \left(\sum_{m=0}^{l} 2 r_{l-m} r_m u \right) z^{-2l-2}, \\
\frac{1}{2} z^2 R^2 &= \sum_{l=0}^{\infty} \left(\sum_{m=0}^{l} \frac{1}{2} r_{l-m} r_m \right) z^{-2l} = \sum_{l=-1}^{\infty} \left(\sum_{m=0}^{l+1} \frac{1}{2} r_{l+1-m} r_m \right) z^{-2l-2}.
\end{aligned}$$

Therefore Equation (A.2) is equivalent to the system of equations consisting of

$$\frac{1}{2} r_0^2 = c_0 \tag{A.3}$$

and

$$\sum_{m=0}^{l}\left(r_{l-m}\,\mathrm{D}_x^2\,r_m-\frac{1}{2}\,\mathrm{D}_x\,r_{l-m}\;\mathrm{D}_x\,r_m-\frac{1}{2}r_{l+1-m}r_m+2r_{l-m}r_m u\right)-\frac{1}{2}r_0 r_{l+1}=c_{l+1}$$

for every $l \geq 0$. Writing the term for $m=0$ separately we get

$$\sum_{m=1}^{l}\left(r_{l-m}\,\mathrm{D}_x^2\,r_m-\frac{1}{2}\,\mathrm{D}_x\,r_{l-m}\;\mathrm{D}_x\,r_m-\frac{1}{2}r_{l+1-m}r_m+2r_{l-m}r_m u\right)$$
$$+\,r_l\,\mathrm{D}_x^2\,r_0-\frac{1}{2}\,\mathrm{D}_x\,r_l\;\mathrm{D}_x\,r_0-\frac{1}{2}r_{l+1}r_0+2r_l r_0 u-\frac{1}{2}r_0 r_{l+1}=c_{l+1}.$$

Solving this equation for r_{l+1} we get an explicit recurrence relation.

$$r_{l+1}=\frac{1}{r_0}\sum_{m=1}^{l}\left(r_{l-m}\;\mathrm{D}_x^2\,r_m-\frac{1}{2}\,\mathrm{D}_x\,r_{l-m}\,\mathrm{D}_x\,r_m-\frac{1}{2}r_{l+1-m}r_m+2r_{l-m}r_m u\right)$$
$$+\,\frac{r_l\,\mathrm{D}_x^2\,r_0}{r_0}-\frac{\mathrm{D}_x\,r_l\;\mathrm{D}_x\,r_0}{2r_0}+2r_l u-\frac{c_{l+1}}{r_0}.$$

Hence once r_0 is fixed, all r_l are uniquely determined. From Equation (A.3) we see that for every C with positive c_0 there exist two solutions to Equation (A.2). If $c_0=0$ there exists exactly one solution. □

We consider Equation (A.2) for $C=\frac{1}{8}$, i.e. $c_0=\frac{1}{8}$ and $c_k=0$ for $k\neq 0$. Let us calculate the first few coefficients of the power series $R=r_0 z^{-1}+r_1 z^{-3}+r_2 z^{-5}+\dots$.

- The first coefficient is $r_0=\pm\sqrt{2c_0}=\pm\frac{1}{2}$. We choose the positive sign: $r_0=\frac{1}{2}$.

  Remark. The choice of sign, and previously the choice of C, are motivated by the fact that with this normalization the highest order term in each KdV equation will have coefficient 1.

- The coefficient of the z^{-2}-term is $r_1=2r_0u=u$.

- The coefficient of the z^{-4}-term is

$$\begin{aligned}r_2&=2\left(r_0\,\mathrm{D}_x^2\,r_1-\frac{1}{2}r_1r_1+2r_0r_1u\right)+2r_1u\\&=2\left(\frac{1}{2}u_{xx}-\frac{1}{2}u^2+u^2\right)+2u^2\\&=u_{xx}+3u^2.\end{aligned}$$

- The coefficient of the z^{-6}-term is

$$\begin{aligned}
r_3 &= 2\left(r_1 \operatorname{D}_x^2 r_1 - \frac{1}{2}(\operatorname{D}_x r_1)^2 - \frac{1}{2}r_2 r_1 + 2r_1^2 u + r_0 \operatorname{D}_x^2 r_2 - \frac{1}{2}r_1 r_2 + 2r_0 r_2 u\right) + 2r_2 u \\
&= 2\left(u u_{xx} - \frac{1}{2}u_x^2 - \frac{1}{2}u r_2 + 2u^3 + \frac{1}{2}\operatorname{D}_x^2 r_2 - \frac{1}{2}u r_2 + u r_2\right) + 2u r_2 \\
&= 2u u_{xx} - u_x^2 + 4u^3 + \operatorname{D}_x^2 r_2 + 2u r_2 \\
&= 2u u_{xx} - u_x^2 + 4u^3 + u_{xxxx} + 6u u_{xx} + 6u_x^2 + 2u u_{xx} + 6u^3 \\
&= u_{xxxx} + 10 u u_{xx} + 5u_x^2 + 10u^3.
\end{aligned}$$

The Korteweg-de Vries hierarchy can be defined as follows.

Definition A.6.

- The *KdV hierarchy* is the family of equations

$$u_t = \operatorname{D}_x r_k[u] \qquad (k \geq 1).$$

- Write $g_k[v] := r_k[v_x]$. The *differentiated potential KdV (DPKdV) hierarchy* is the family of equations

$$v_{xt} = \operatorname{D}_x g_k[v] \qquad (k \geq 1).$$

- The *potential KdV (PKdV) hierarchy* is the family of equations

$$v_t = g_k[v] \qquad (k \geq 1).$$

Proposition A.7. *The r_k satisfy* $\dfrac{\delta r_k}{\delta u} = (4k-2)\, r_{k-1}$.

Proof. In this proof we use a prime, $'$, as well as D_x, to denote the total x-derivative. We apply the vertical exterior derivative δ to Equation (A.2):

$$R''\delta R + R\delta R'' - R'\delta R' + 4\left(u - \frac{1}{4}z^2\right) R\delta R + 2R^2\delta u = 0.$$

Write $\zeta := z^2$ and $R_\zeta := \frac{\partial R}{\partial \zeta}$. Multiply the last equation by $\frac{R_\zeta}{2R^2}$ to obtain

$$\begin{aligned}
0 &= \frac{R_\zeta R''\delta R}{2R^2} + \frac{R_\zeta \delta R''}{2R} - \frac{R_\zeta R'\delta R'}{2R^2} + 2\left(u - \frac{1}{4}z^2\right)\frac{R_\zeta \delta R}{2R} + R_\zeta \delta u \\
&= \left(\frac{R_\zeta R''}{2R^2} + 2\left(u - \frac{1}{4}z^2\right)\frac{R_\zeta}{R}\right)\delta R + \operatorname{D}_x\left(\frac{R_\zeta \delta R'}{2R}\right) - \frac{R_\zeta' \delta R'}{2R} + R_\zeta \delta u. \qquad \text{(A.4)}
\end{aligned}$$

Deriving Equation (A.2) with respect to ζ we get

$$R_\zeta R'' + RR''_\zeta - R_x R'_\zeta + 4\left(u - \frac{1}{4}z^2\right) RR_\zeta - \frac{1}{2}R^2 = 0,$$

and thus

$$\frac{R_\zeta R''}{2R^2} + \frac{R''_\zeta}{2R} - \frac{R' R'_\zeta}{2R^2} + \frac{2\left(u - \frac{1}{4}z^2\right) R_\zeta}{R} = \frac{1}{4}. \tag{A.5}$$

Combining Equations (A.4) and (A.5) we arrive at

$$\begin{aligned}\frac{1}{4}\delta R &= -R_\zeta \delta u - \mathrm{D}_x\left(\frac{R_\zeta \delta R'}{2R}\right) + \frac{R'_\zeta \delta R'}{2R} + \frac{R''_\zeta \delta R}{2R} - \frac{R' R'_\zeta \delta R}{2R^2}\\ &= -R_\zeta \delta u - \mathrm{D}_x\left(\frac{R_\zeta \delta R'}{2R} + \frac{R'_\zeta \delta R}{2R}\right).\end{aligned}$$

After (formally) integrating this becomes

$$\int \delta R = \int -4R_\zeta \delta u.$$

It follows that $\frac{\delta R}{\delta u} = -4R_\zeta$, so

$$\sum_{k=0}^{\infty} \frac{\delta r_k}{\delta u}\zeta^{-k-\frac{1}{2}} = \sum_{k=0}^{\infty} 4\left(k + \frac{1}{2}\right) r_k \zeta^{-k-\frac{3}{2}} = \sum_{k=1}^{\infty} (4k-2)\, r_{k-1}\zeta^{-k-\frac{1}{2}}.$$

Therefore

$$\frac{\delta r_k}{\delta u} = (4k-2)\, r_{k-1}.$$

□

Corollary A.8. *Set* $h_k := \frac{1}{4k+2} g_{k+1}$*, then the* g_k *and* h_k *satisfy*

$$\frac{\delta g_k}{\delta v_x} = (4k-2)\, g_{k-1} \qquad \text{and} \qquad \frac{\delta h_k}{\delta v_x} = g_k.$$

Theorem A.9. *The functions*

$$L_k := \frac{1}{2} v_x v_t - h_k$$

are Lagrangians for the DPKdV equations.

Proof. Literally the same as the proof of Theorem 4.13. □

A.4. Table of explicitly computed quantities for the KdV hierarchy

	$i = 1$	$i = 2$	$i = 3$
r_i	u	$u_{xx} + 3u^2$	$u_{x^4} + 10uu_{xx} + 5u_x^2 + 10u^3$
$\mathrm{D}_x\, r_i$	u_x	$u_{xxx} + 6uu_x$	$u_{x^5} + 20u_x u_{xx} + 10uu_{xxx} + 30u^2 u_x$
g_i	v_x	$v_{xxx} + 3v_x^2$	$v_{x^5} + 10v_x v_{xxx} + 5v_{xx}^2 + 10v_x^3$
$\mathrm{D}_x\, g_i$	v_{xx}	$v_{x^4} + 6v_x v_{xx}$	$v_{x^6} + 20v_{xx}v_{xxx} + 10v_x v_{x^4} + 30v_x^2 v_{xx}$
h_i	$\frac{v_{xxx}}{6} + \frac{1}{2}v_x^2$	$\frac{1}{10}v_{x^5} + \frac{1}{2}v_{xx}^2 + v_x v_{xxx} + v_x^3$	$\frac{1}{14}v_{x^7} + v_x v_{x^5} + 2v_{xx}v_{x^4} + \frac{3}{2}v_{xxx}^2 + 5v_x^2 v_{xxx} + 5v_x v_{xx}^2 + \frac{5}{2}v_x^4$

a_{23}	$3v_{t_3}v_x^2 + v_x v_{t_3xx} + v_{t_3}v_{xxx} + \frac{1}{10}v_{t_3x^4}$
a_{32}	$10v_{t_2}v_x^3 + 5v_{t_2}v_{xx}^2 + 5v_x^2 v_{t_2xx} + 10v_{t_2}v_x v_{xxx} + 2v_{t_2xx}v_{xxx} + v_{xx}v_{t_2xxx} + v_x v_{t_2x^4} + v_{t_2}v_{x^5} + \frac{1}{14}v_{t_2x^6}$
b_{23}	$201v_x^5 + 960v_x^2 v_{xx}^2 + 640v_x^3 v_{xxx} + 692v_{xx}^2 v_{xxx} + 569v_x v_{xxx}^2 + 762v_x v_{xx} v_{x^4} + 104v_{x^4}^2 + 189v_x^2 v_{x^5} + 171v_{xxx}v_{x^5} + 81v_{xx}v_{x^6} + 27v_x v_{x^7} + \frac{3}{2}v_{x^9}$
b_{32}	$207v_x^5 + 945v_x^2 v_{xx}^2 + 660v_x^3 v_{xxx} + 699v_{xx}^2 v_{xxx} + 575v_x v_{xxx}^2 + 750v_x v_{xx} v_{x^4} + 103v_{x^4}^2 + 192v_x^2 v_{x^5} + 172v_{xxx}v_{x^5} + 81v_{xx}v_{x^6} + 27v_x v_{x^7} + \frac{3}{2}v_{x^9}$

Definitions and usage

- Schrödinger operator $\mathscr{L} := \partial^2 + u$
- $r_k := 2^{2k-1} \operatorname{res}\left(\mathscr{L}^{\frac{2k-1}{2}}\right)$
- KdV: $u_{t_i} = \mathrm{D}_x\, r_i[u]$
- $g[v] := r[v_x]$
- DPKdV: $v_{xt_i} = \mathrm{D}_x\, g_i[v]$
- PKdV: $v_{t_i} = g_i[v]$
- Lagrangians for DPKdV: $L_i := L_{1i} := \frac{1}{2} v_x v_{t_i} - h_i$, where $h_i := \frac{1}{4i+2} g_{i+1}$
- $a_{ij} := v_{t_j} \frac{\delta_1 h_i}{\delta v_x} + v_{xt_j} \frac{\delta_1 h_i}{\delta v_{xx}} + v_{xxt_j} \frac{\delta_1 h_i}{\delta v_{xxx}} + \dots$
- $b_{i0} := \frac{1}{2} g_i, \qquad b_{i\,j+1} := b_{i+1\,j} + g_{j+1} g_i - \mathrm{D}_x^2\, g_j\, g_i + \mathrm{D}_x\, g_j\; \mathrm{D}_x\, g_i - g_j\, \mathrm{D}_x^2\, g_i - 4 v_x g_j g_i$
- Lagrangian two-form: $\mathcal{L} = \sum_{i<j} L_{ij}\, \mathrm{d}t_i \wedge \mathrm{d}t_j$, where, for $i, j > 1$,

  $L_{ij} := \frac{1}{2}(v_{t_i} g_j - g_i v_{t_j}) + \left(a_{ij} - a_{ji}\right) - \frac{1}{2}(b_{ij} - b_{ji})$

B. Zusammenfassung in deutscher Sprache

Ich glaube nicht, dass es irgendetwas auf der ganzen Welt gibt, was man in Berlin nicht lernen könnte - außer der deutschen Sprache!

Mark Twain[1]

In der klassischen Mechanik sind die Hamiltonsche und Lagrangesche Formulierung äquivalent. Im Bereich der integrierbaren Systeme ist die Lage ganz anders. Manche der wichtigsten Definitionen von Integrierbarkeit benutzen einen Hamiltonschen Gesichtspunkt, aber eine allgemein akzeptierte Lagrangesche Theorie von integrierbaren Systemen gibt es im Moment nicht. Einen Vorschlag für solch ein Konzept wird in dieser Arbeit entwickelt.

Gegeben $N-1$ kommutierende Lagrangesche Flüsse, können wir unterschiedliche Zeitkoordinaten t_i für jeden einzelnen Fluss einführen und gleichzeitige Lösungen der Euler-Lagrange-Gleichungen als Funktionen $u : \mathbb{R}^N \to \mathbb{R} : (x, t_2, \dots, t_N) \mapsto u(x, t_2, \dots, t_N)$ betrachten. Das übliche Lagrangesche Kriterium ist in diesem Kontext, dass die Wirkung $\int_S \mathcal{L}$ auf jeder (x, t_i)-Ebene S stationär ist. In einem *Pluri-Lagrangeschen System* gilt ein viel stärkeres Kriterium: die Wirkung muss auf jeder zwei-dimensionalen Mannigfaltigkeit S in $\mathbb{R}^N$ stationär sein.

Dieser Ansatz scheint vielleicht unmotiviert, aber ähnliche Ideen sind in anderen Gebieten der Mathematik gängig. Ein Beispiel dafür ist die Theorie der *pluriharmonischen Funktionen*. Eine Funktion f heißt pluriharmonisch, wenn das Dirichletfunktional $\int_\Gamma \left| \frac{\partial (f \circ \Gamma)}{\partial z} \right|^2 dz \wedge d\bar{z}$ für jede holomorphe Kurve $\Gamma : \mathbb{C} \to \mathbb{C}^m$ minimal ist. Andere Beispiele für verwandte Ideen sind *Baxters Z-Invariante* in der statistischen Mechanik und das klassische Konzept von Variationssymmetrien.

Die Absicht dieser Arbeit ist eine Pluri-Lagrangesche Theorie für integrierbare Hierarchien von Differentialgleichungen zu entwickeln. Es gibt drei Hauptziele: erstens die Herleitung der *Multi-Zeit-Euler-Lagrange-Gleichungen*, zweitens die Konstruktion einer pluri-Lagrangesche Form für die Korteweg-de-Vries-Hierarchie und drittens eine kurze Besprechung der Beziehung zwischen pluri-Lagrangeschen Systemen und der bekannten Hamiltonschen Theorie der integrierbaren Systeme.

[1] Original: "I don't believe there is anything in the whole earth that you can't learn in Berlin except the German language."

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