A variational principle for integrable systems

Mats Vermeeren

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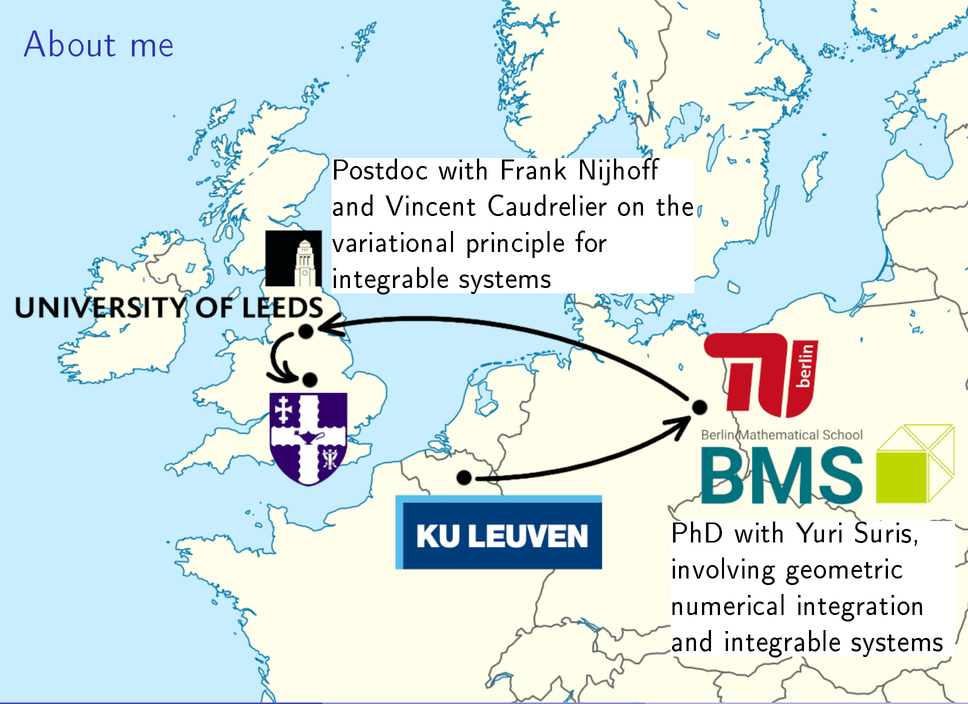
About me



Postdoc with Frank Nijhoff and Vincent Caudrelier on the variational principle for integrable systems

PhD with Yuri Suris, involving geometric numerical integration and integrable systems

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Poisson Brackets

Poisson bracket of two functions on T^*Q:

$$\{f,g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

Dynamics of a Hamiltonian system:

$$\dot{q}_i = \{q_i, H\}, \qquad \dot{p}_i = \{p_i, H\}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} f(q,p) = \{f(q,p), H\}$$

In particular: f is conserved if and only if $\{f,H\} = 0$.

Properties:

$$\begin{aligned}
\text{anti-symmetry:}\quad & \{f,g\} = -\{g,f\} \\
\text{bilinearity:}\quad & \{f, g+\lambda h\} = \{f,g\} + \lambda\{f,h\} \\
\text{Leibniz property:}\quad & \{f, gh\} = \{f,g\}h + g\{f,h\} \\
\text{Jacobi identity:}\quad & \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0
\end{aligned}$$

Liouville-Arnold integrability

A Hamiltonian system with Hamilton function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville-Arnold integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \ldots H_N$ such that $\{H_i, H_j\} = 0$.

- Each H_i defines a dynamical system.
- Each H_i is a conserved quantity for all these systems.
- The dynamics is confined to a leaf of the foliation $\{H_i = \text{const}\}$.
- There exists a symplectic change of variables $(p, q) \mapsto (\bar{p}, \bar{q})$ such that $H_i(p, q) = \bar{H}_i(\bar{p})$.

  System evolves linearly in these action-angle variables.
- The flows commute...

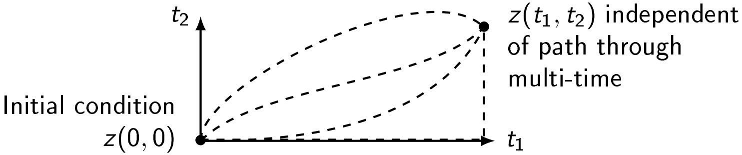
Multi-time perspective on a Liouville integrable system

Let $z = (q, p)$. Consider two Hamiltonian ODEs

$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_1} = \{f(z), H_1(z)\}$$
$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_2} = \{f(z), H_2(z)\}$$

with $\{H_1, H_2\} = 0$

The flows commute, meaning that evolution can be parametrised by the (t_1, t_2) plane, called multi-time.



Additional commuting equations can be accommodated by increasing the dimension of multi-time: $\mathbb{R}^n$ instead of $\mathbb{R}^2$.

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Lagrangian formulation of Liouville integrable system

On the Hamiltonian side, commutativity is implied by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Suppose we have Lagrange functions L_i associated to H_i.

Lagrangian multi-form (Pluri-Lagrangian) principle for ODEs

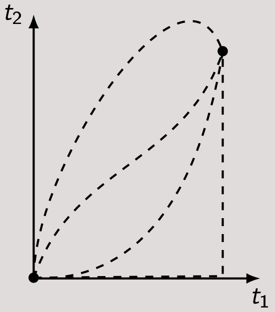
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q]\,\mathrm{d}t_i.$$

Look for dynamical variables $q(t_1, \ldots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$



Multi-time Euler-Lagrange equations for $\mathcal{L} = \sum_i L_i[q]\,\mathrm{d}t_i$

Usual Euler-Lagrange equations: $\dfrac{\delta_i L_i}{\delta q_I} = 0 \qquad \forall I \not\ni t_i,$

Additional conditions: $\dfrac{\delta_i L_i}{\delta q_{It_i}} = \dfrac{\delta_j L_j}{\delta q_{It_j}} \qquad \forall I,$

where

- I is a multi-index, q_I the corresponding partial derivative
- $\dfrac{\delta_i}{\delta q_I}$ is the variational derivative in the direction of t_i:

$$\frac{\delta_i L_i}{\delta q_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial L_i}{\partial q_{It_i^\alpha}}$$
$$= \frac{\partial L_i}{\partial q_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{It_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_i}{\partial q_{It_i^2}} - \dots$$

Example: Kepler Problem

The classical Lagrangian

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e \qquad (e \text{ fixed unit vector})$$

into a Lagrangian 1-form $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$.

Multi-time Euler-Lagrange equations:

$$\frac{\delta_1 L_1}{\delta q} = 0 \quad \Rightarrow \quad q_{t_1 t_1} = -\frac{q}{|q|^3} \qquad \text{(Keplerian motion)}$$

$$\frac{\delta_2 L_2}{\delta q} = 0 \quad \Rightarrow \quad q_{t_1 t_2} = e \times q_{t_1}$$

$$\frac{\delta_2 L_2}{\delta q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_2} = e \times q \qquad \text{(Rotation)}$$

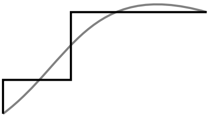
$$\frac{\delta_1 L_1}{\delta q_{t_1}} = \frac{\delta_2 L_2}{\delta q_{t_2}} \quad \Rightarrow \quad q_{t_1} = q_{t_1}$$

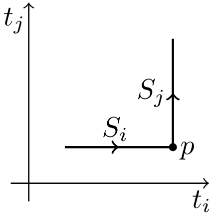
Derivation of the multi-time Euler-Lagrange equations

Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[q]\,\mathrm{d}t_i$

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.



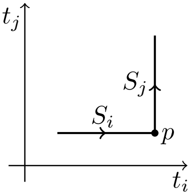


Variations are local, so it is sufficient to look at a general L-shaped curve $S = S_i \cup S_j$.

Derivation of the multi-time Euler-Lagrange equations

$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \sum_{I \not\ni t_i} \sum_{\alpha=0}^{\infty} \frac{\partial L_i}{\partial q_{It_i^\alpha}} \delta q_{It_i^\alpha} \,\mathrm{d}t_i$$

$$= \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta q_I} \delta q_I \,\mathrm{d}t_i + \sum_I \frac{\delta_i L_i}{\delta q_{It_i}} \delta q_I \bigg|_p ,$$



where

$$\frac{\delta_i L_i}{\delta q_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial L_i}{\partial q_{It_i^\alpha}} = \frac{\partial L_i}{\partial q_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{It_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_i}{\partial q_{It_i^2}} - \dots$$

$$\frac{\delta_i L_i}{\delta q_I} = 0 \qquad \forall I \not\ni t_i \qquad \text{and} \qquad \frac{\delta_i L_i}{\delta q_{It_i}} = \frac{\delta_j L_j}{\delta q_{It_j}} \qquad \forall I$$

Suris. Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geometric Mechanics, 2013

Suris, V. On the Lagrangian structure of integrable hierarchies. In: Advances in Discrete Differential Geometry, Springer. 2016.

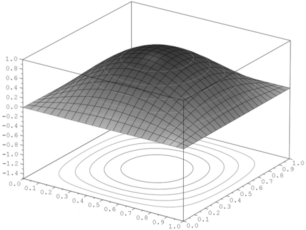
Pluri-Lagrangian principle for PDEs ($d = 2$)

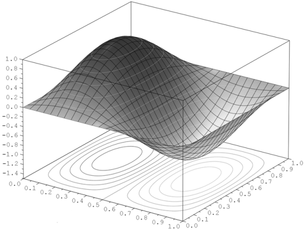
Notation: for PDEs we use u instead of q for the field.

Given a 2-form

$$\mathcal{L}[u] = \textstyle\sum_{i,j} L_{ij}[u]\, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $u : \mathbb{R}^N \to \mathbb{R}$, such that $\displaystyle\int_\Gamma \mathcal{L}[u]$ is critical on all smooth 2-surfaces Γ in multi-time $\mathbb{R}^N$, w.r.t. variations of u.





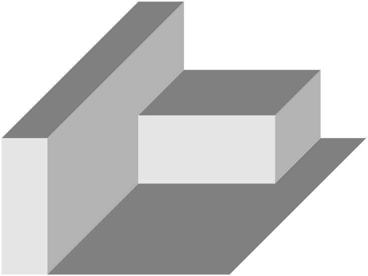
Example: KdV hierarchy, where $t_1 = x$ is the shared space coordinate, t_i time for i-th flow. (Details to follow.)

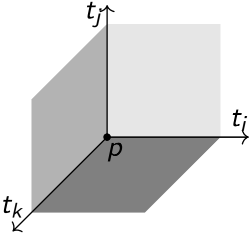
Multi-time EL equations

Consider a Lagrangian 2-form $\mathcal{L}[u] = \sum_{i,j} L_{ij}[u]\,\mathrm{d}t_i \wedge \mathrm{d}t_j$.

Every smooth surface can be approximated arbitrarily well by stepped surfaces.

It is sufficient to require criticality on stepped surfaces. Variations can be taken locally, so it is sufficient to consider elementary corners.





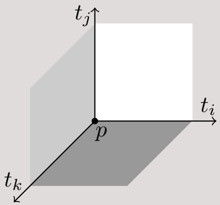
Multi-time EL equations

for $\mathcal{L}[u] = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial u_{I t_i^\alpha t_j^\beta}}$$

Example: Potential KdV hierarchy

$u_{t_2} = Q_2 = u_{xxx} + 3u_x^2,$
$u_{t_3} = Q_3 = u_{xxxxx} + 10u_x u_{xxx} + 5u_{xx}^2 + 10u_x^3,$
where we identify $t_1 = x$.

The differentiated equations $u_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x} Q_i$ are Lagrangian with

$$L_{12} = \frac{1}{2} u_x u_{t_2} - \frac{1}{2} u_x u_{xxx} - u_x^3,$$
$$L_{13} = \frac{1}{2} u_x u_{t_3} - \frac{1}{2} u_{xxx}^2 + 5 u_x u_{xx}^2 - \frac{5}{2} u_x^4.$$

A suitable coefficient L_{23} of

$$\mathcal{L} = L_{12}\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}\,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

can be found (nontrivial task!) in the form

$$L_{23} = \frac{1}{2}(u_{t_3} Q_2 - u_{t_2} Q_3) + p_{23}.$$

Example: Potential KdV hierarchy

- The equations $\dfrac{\delta_{12} L_{12}}{\delta u} = 0$ and $\dfrac{\delta_{13} L_{13}}{\delta u} = 0$ yield
$$u_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x} Q_2 \qquad \text{and} \qquad u_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x} Q_3 .$$
- The equations $\dfrac{\delta_{12} L_{12}}{\delta u_x} = \dfrac{\delta_{32} L_{32}}{\delta u_{t_3}}$ and $\dfrac{\delta_{13} L_{13}}{\delta u_x} = \dfrac{\delta_{23} L_{23}}{\delta u_{t_2}}$ yield
$$u_{t_2} = Q_2 \qquad \text{and} \qquad u_{t_3} = Q_3 ,$$
the evolutionary equations!
- All other multi-time EL equations are corollaries of these.

Suris, V. On the Lagrangian structure of integrable hierarchies. In: Advances in Discrete Differential Geometry, Springer. 2016.

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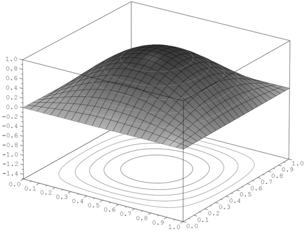
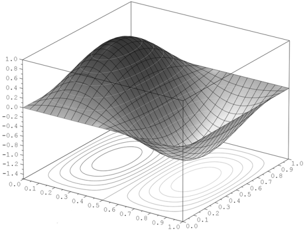
Closedness of the Lagrangian form

One could require additionaly that $\mathcal{L}$ is closed on solutions

$\hookrightarrow$ "Lagrangian multiform systems" (Leeds).

Then the action is not just critical on every curve/surface, but also takes the same value on every curve/surface.





Maybe this is not necessary as part of the definition, because one can show

Proposition

$\mathrm{d}\mathcal{L}$ is constant on the set of solutions.

$\hookrightarrow$ "Pluri-Lagrangian systems" (Berlin).

Closedness: examples

Kepler problem $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$ with

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e \qquad (e \text{ fixed unit vector})$$

Multi-time Euler-Lagrange equations: $q_{t_1 t_1} = -\dfrac{q}{|q|^3}$ and $q_{t_2} = e \times q$

Coefficient of $\mathrm{d}\mathcal{L}$:

$$\frac{\mathrm{d}L_2}{\mathrm{d}t_1} - \frac{\mathrm{d}L_1}{\mathrm{d}t_2} = \left(q_{t_1 t_1} + \frac{q}{|q|^3} \right) (q_{t_2} - e \times q)$$

Potential KdV $u_{t_2} = Q_2$ and $u_{t_3} = Q_3$. Coefficient of $\mathrm{d}\mathcal{L}$:

$$\begin{aligned} \frac{\mathrm{d}L_{23}}{\mathrm{d}x} - \frac{\mathrm{d}L_{13}}{\mathrm{d}t_2} + \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} &= \frac{1}{2} (u_{t_2} - Q_2) \frac{\mathrm{d}}{\mathrm{d}x} (u_{t_3} - Q_3) \\ &\quad - \frac{1}{2} (u_{t_3} - Q_3) \frac{\mathrm{d}}{\mathrm{d}x} (u_{t_2} - Q_2) \end{aligned}$$

$\delta\mathrm{d}\mathcal{L}$

In the previous examples, the coefficients of $\mathrm{d}\mathcal{L}$ have a double zero on solutions.

This is no coincidence:

Theorem

The variational principle is equivalent to

$$\delta\mathrm{d}\mathcal{L} = 0,$$

i.e. the exterior derivative is invariant under infinitesimal variations.

If we do not require that $\delta\mathcal{L} = 0$ on solutions (the "pluri-Lagrangian" convention), then it is possible that

$$\mathrm{d}\mathcal{L} = \text{constant} + \text{double zero}.$$

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Closedness and involutivity

We can pass between the pluri-Lagrangian and Hamiltonian formalisms for 1-forms* and 2-forms†.

Lemma ($\mathrm{d}\mathcal{L}$ for 1-forms)

On solutions there holds $\dfrac{\mathrm{d}L_j}{\mathrm{d}t_i} - \dfrac{\mathrm{d}L_i}{\mathrm{d}t_j} = \{H_j, H_i\}$.

It follows that:

Theorem

The Hamiltonians are in involution if and only if $\mathrm{d}\mathcal{L} = 0$ on solutions.

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*Suris. Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geometric Mechanics, 2013

†V. Hamiltonian structures for integrable hierarchies of Lagrangian PDEs Open Communications in Nonlinear Mathematical Physics, 2021.

Variational Symmetries and Lagrangian forms

Connection provided by the closedness property $\mathrm{d}\mathcal{L} = 0$:

1-forms If $\mathrm{d}\left(\sum_i L_i \, \mathrm{d}t_i \right) = 0$, then $\dfrac{\mathrm{d}L_k}{\mathrm{d}t_j} = \dfrac{\mathrm{d}L_j}{\mathrm{d}t_k}$

$\Rightarrow$ t_j-flow changes L_k by a t_k-derivative.

$\Rightarrow$ flows are variational symmetries of each other:

2-forms If $\mathrm{d}\left(\sum_{i,j} L_{ij} \, \mathrm{d}t_i \wedge \mathrm{d}t_j \right) = 0$, then $\dfrac{\mathrm{d}L_{ij}}{\mathrm{d}t_k} = \dfrac{\mathrm{d}L_{ik}}{\mathrm{d}t_j} - \dfrac{\mathrm{d}L_{jk}}{\mathrm{d}t_i}$

$\Rightarrow$ t_k-flow changes L_{ij} by a divergence in (t_i, t_j).

$\Rightarrow$ flows are variational symmetries of each other

Idea: use variational symmetries to construct Lagrangian form.

Sleigh, Nijhoff, Caudrelier. Variational symmetries and Lagrangian multiforms. Letters in Mathematical Physics, 2020.

Petrera, V. Variational symmetries and pluri-Lagrangian structures for integrable hierarchies of PDEs. European Journal of Mathematics, 2021.

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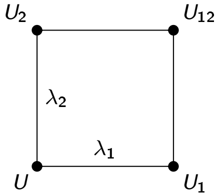
Discretisation of Hamiltonian systems

Hamiltonian ODE	$\rightarrow$	symplectic map
Liouville-Arnold system	$\rightarrow$	commuting symplectic maps (or symplectic map with conserved quantities?)
Hamiltonian PDE	$\rightarrow$	partial difference equation: multisymplectic map on a lattice?

Quad equations

$$\mathcal{Q}(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2) = 0$$

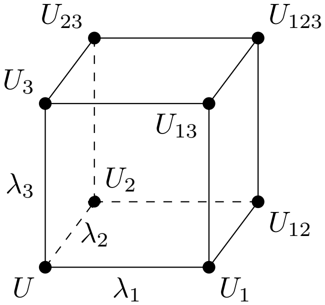
Subscripts of U denote lattice shifts, λ_1, λ_2 are parameters.
Invariant under symmetries of the square, affine in each of U, U_1, U_2, U_{12}.



Integrability for systems quad equations:
Multi-dimensional consistency of

$$\mathcal{Q}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) = 0,$$

i.e. the thrunderee ways of calculating U_{123} give the same result.



Classification (under some extra assumptions) by Adler, Bobenko and Suris (ABS List).

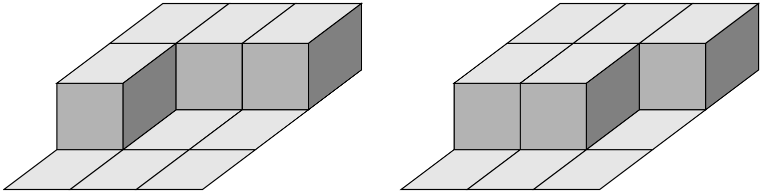
Example: lattice potential KdV:
$(U - U_{12})(U_1 - U_2) - \lambda_1 + \lambda_2 = 0$

Variational principle for quad equations

For some discrete 2-form

$$\mathcal{L}(\Box_{ij}) = \mathcal{L}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j),$$

the action $\sum_{\Box \in \Gamma} \mathcal{L}(\Box)$ is critical on all 2-surfaces Γ in $\mathbb{Z}^N$ simultaneously.



Discretising Hamiltonian structures was ambiguous. Here, the discrete and continuous variational principles are essentially the same.

Lobb, Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009.

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Continuum limit of an integrable difference equation

Miwa shifts*

Skew embedding of the mesh $\mathbb{Z}^N$ into multi-time $\mathbb{R}^N$
Discrete U is a sampling of the continuous u:

$$U = U(\mathrm{n}) = u(t_1, t_2, \dots, t_N),$$

$$U_i = U(\mathrm{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \dots, t_N + 2(-1)^N \frac{\lambda_i^N}{N} \right)$$

Write quad equation in terms of q and expand in λ_1.

In the leading order, we only see t_1-derivatives of q, but we want to obtain PDEs.

$\hookrightarrow$ leading order cancellation required to get a meaningful result.

$\hookrightarrow$ whole hierarchy from single difference equation.

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*Miwa. On Hirota's difference equations. Proceedings of the Japan Academy A, 1982.

Continuum limit of the Lagrangian

- Using Miwa correspondence:

$$\text{Discrete } L \quad \to \quad \text{Power series } \mathcal{L}_{\mathrm{disc}}[u(\mathrm{t})]$$

Action for $\mathcal{L}_{\mathrm{disc}}[u(\mathrm{t})]$ is still a sum.

- Euler-Maclaurin formula (sum $\xleftrightarrow{\text{formal power series}}$ integral)

$$\mathcal{L}_{\mathrm{Miwa}}([u],\lambda_1,\lambda_2) = \sum_{i,j=0}^{\infty} \frac{B_i B_j}{i!j!} \partial_1^i \partial_2^j \mathcal{L}_{\mathrm{disc}}([u],\lambda_1,\lambda_2).$$

where the differential operators are $\quad \partial_k = \sum_{j=1}^{N} (-1)^{j+1} \frac{2\lambda_k^j}{j} \frac{\mathrm{d}}{\mathrm{d}t_j}$

- Then there holds $L_{\mathrm{disc}}(\square) = \displaystyle\int_{\blacksquare} \mathcal{L}_{\mathrm{Miwa}}([u(\mathrm{t})],\lambda_1,\lambda_2)\, \eta_1 \wedge \eta_2,$

where η_1 and η_2 are the 1-forms dual to the Miwa shifts.

This suggests the Lagrangian 2-form

$$\sum_{1 \leq i < j \leq N} \mathcal{L}_{\mathrm{Miwa}}([u],\lambda_i,\lambda_j)\, \eta_i \wedge \eta_j.$$

Continuum limit of a Lagrangian 2-form

$L(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2)$ Suitable choice $\Rightarrow$ leading order cancellat

$$\Big\downarrow \quad \text{Miwa shifts, Taylor expansion}$$

$$\mathcal{L}_{\mathrm{disc}}([u], \lambda_1, \lambda_2)$$

$$\Big\downarrow \quad \text{Euler-Maclaurin formula}$$

$$\mathcal{L}_{\mathrm{Miwa}}([u], \lambda_1, \lambda_2) = \sum_{i,j=1}^{\infty} (-1)^{i+j} 4 \frac{\lambda_1^i}{i} \frac{\lambda_2^j}{j} \mathcal{L}_{i,j}[u]$$

$$\Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow$$

$$\sum_{1 \leq i < j \leq N} \mathcal{L}_{\mathrm{Miwa}}([u], \lambda_i, \lambda_j)\, \eta_i \wedge \eta_j \approx \sum_{1 \leq i < j \leq N} \mathcal{L}_{i,j}[u]\, \mathrm{d}t_i \wedge \mathrm{d}t_j$$

V. Continuum limits of pluri-Lagrangian systems. Journal of Integrable Systems, 2019.

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Summary

- The pluri-Lagrangian (or Lagrangian multiform) principle is a widely applicable characterization of integrability:

  Applies to ODEs and PDEs, discrete and continuous.
- Closedness of the Lagrangian form, i.e. $\mathrm{d}\mathcal{L} = 0$, is related to variational symmetries and Hamiltonians in involution.
- Tools to construct Lagrangian 1- and 2-forms:
  - Variational symmetries
  - Hamiltonian structures
  - Continuum limits
  - ...

To do

Work in progress:

- A non-abelian symmetry group can be captured by using a Lie group as multi-time instead of $\mathbb{R}^N$.
- Application to semi-discrete systems.

Further questions:

- Relation to bi-Hamiltonian structures
- Use the pluri-Lagrangian principle to characterise special solutions.
- Classification of Lagrangian multi-forms.
- Application to infinite-dimensional symmetry groups

  $\hookrightarrow$ Noether's second theorem.
- Application to quantum integrable systems, path integrals, ...
- ...

Selected references

Discrete	▶ Lobb, Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009. ▶ Boll, Petrera, Suris. What is integrability of discrete variational systems? Proc. R. Soc. A. 2014. ▶ V. Continuum limits of pluri-Lagrangian systems. Journal of Integrable Systems, 2019
Continuous	▶ Suris. Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geometric Mechanics, 2013 ▶ Suris, V. On the Lagrangian structure of integrable hierarchies. In: Advances in Discrete Differential Geometry, Springer. 2016. ▶ Petrera, V. Variational symmetries and pluri-Lagrangian structures for integrable hierarchies of PDEs. European Journal of Mathematics, 2021 ▶ V. Hamiltonian structures for integrable hierarchies of Lagrangian PDEs Open Communications in Nonlinear Mathematical Physics, 2021.

Thank you for your attention!