Loughborough University

A Lagrangian perspective on integrability

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Contents

Hamiltonian Systems

Hamilton function $H : T^*Q \cong \mathbb{R}^{2N} \to \mathbb{R} : (q, p) \mapsto H(q, p)$ determines dynamics:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Poisson bracket of two functionals on T^*Q:

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$$

Dynamics:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{q}, \mathrm{p}) = \{H(q, p), f(\mathrm{q}, \mathrm{p})\}$$

Background: Liouville integrability

A Hamiltonian system with Hamilton function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \dots H_N$ such that $\{H_i, H_j\} = 0$.

- Each H_i defines its own flow: N dynamical systems
- Each H_i is a conserved quantity for all flows.
- The dynamics is confined to a leaf of the foliation $\{H_i = \text{const}\}$.
- If this foliation is compact, its leaves are tori.
- Dynamics on these tori are linear in action-angle variables.
- The flows commute.

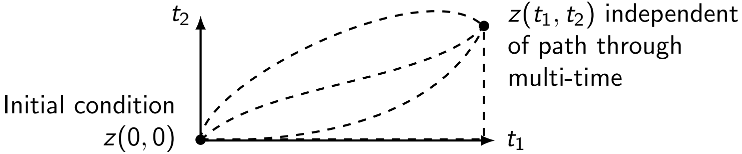
Integrability = being part of a large set of compatible equations.

Two commuting flows

Let $z = (q, p)$. Consider two Hamiltonian ODEs

$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_1} = \{H_1(z), f(z)\}$$
$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_2} = \{H_2(z), f(z)\}$$
with $\{H_1, H_2\} = 0$

The flows commute, meaning that evolution can be parametrised by the (t_1, t_2) plane, called multi-time.



Additional commuting equations can be accommodated by increasing the dimension of multi-time: $\mathbb{R}^n$ instead of $\mathbb{R}^2$.

Lagrangian formulation of Liouville integrable system

On the Hamiltonian side, commutativity is implied by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Suppose we have Lagrange functions L_i associated to H_i.

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Pluri-Lagrangian (Lagrangian multi-form) principle for ODEs

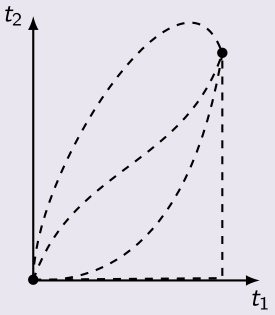
combine the L_i into a Lagrangian 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q] \, \mathrm{d}t_i.$$

Look for dynamical variables $q(t_1, \ldots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$



Multi-time Euler-Lagrange equations

Assume that

$$L_1[q] = L_1(q, q_{t_1}) \qquad \text{and} \qquad L_i[q] = L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1$$

Then the multi-time Euler-Lagrange equations for $\mathcal{L} = \sum_i L_i[q]\,\mathrm{d}t_i$ are

Usual Euler-Lagrange equations: $$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_{t_i}} = 0$$

Usual EL wrt to alien derivatives: $$\frac{\partial L_i}{\partial q_{t_1}} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_{t_1 t_i}} = 0, \quad i \neq 1$$

Additional conditions: $$\frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}}$$

Suris. Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geometric Mechanics, 2013

Generalisation to higher order Lagrangians in:

Suris, V. On the Lagrangian structure of integrable hierarchies. In: Advances in Discrete Differential Geometry, Springer. 2016.

Example: Kepler Problem

The classical Lagrangian

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e \qquad (e \text{ fixed unit vector})$$

into a Lagrangian 1-form $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$.

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Multi-time Euler-Lagrange equations:

$$\frac{\partial L_1}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L_1}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_1 t_1} = -\frac{q}{|q|^3} \qquad \text{(Keplerian motion)}$$

$$\frac{\partial L_2}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_2}\frac{\partial L_2}{\partial q_{t_2}} = 0 \quad \Rightarrow \quad q_{t_1 t_2} = e \times q_{t_1}$$

$$\frac{\partial L_2}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_2} = e \times q \qquad \text{(Rotation)}$$

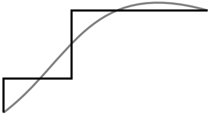
$$\frac{\partial L_1}{\partial q_{t_1}} = \frac{\partial L_2}{\partial q_{t_2}} \quad \Rightarrow \quad q_{t_1} = q_{t_1}$$

Derivation of the multi-time Euler-Lagrange equations

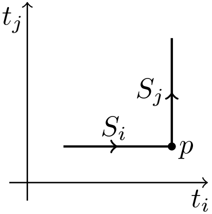
Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[q] \, \mathrm{d}t_i$

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.



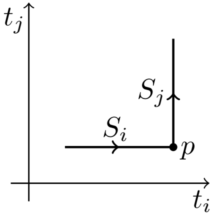
Variations are local, so it is sufficient to look at an L-shaped curve $S = S_i \cup S_j$.



Derivation of the multi-time Euler-Lagrange equations

On one of the straight pieces, S_i ($i \neq 1$), we get

$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \left(\frac{\partial L_i}{\partial q} \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} + \frac{\partial L_i}{\partial q_{t_i}} \delta q_{t_i} \right) \mathrm{d}t_i$$



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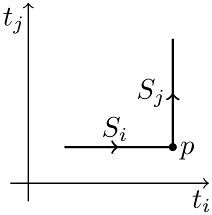

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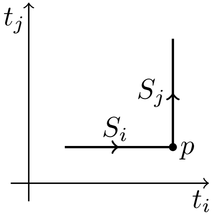
Integration by parts (wrt t_i only) yields

$$\delta \int_{S_i} L_i \, \mathrm{d}t_i = \int_{S_i} \left(\left(\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} \right) \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} \right) \mathrm{d}t_i + \frac{\partial L_i}{\partial q_{t_i}} \delta q \bigg|_p$$

Since p is an interior point of the curve, we cannot set $\delta q(p) = 0$!

Derivation of the multi-time Euler-Lagrange equations



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Arbitrary δq and δq_{t_1} so we find:

Multi-time Euler-Lagrange equations

$$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} = 0, \qquad \frac{\partial L_i}{\partial q_1} = 0, \qquad \frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}}$$

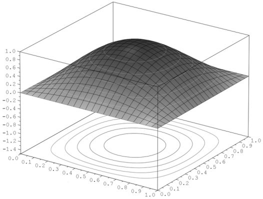
PDEs (2-dimensional)

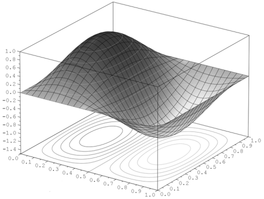
Pluri-Lagrangian (Lagrangian multi-form) principle

Given a 2-form

$$\mathcal{L} = \sum_{i,j} L_{ij}[q] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $q(t_1, \ldots, t_N)$, such that $\displaystyle\int_\Gamma \mathcal{L}$ is critical on all smooth 2-dimensional surfaces Γ in multi-time $\mathbb{R}^N$, w.r.t. variations of q.





Multi-time Euler-Lagrange equations are again a combination of the usual Euler-Lagrange equations and new ones involving several L_{ij}.

Examples: potential KdV hierarchy, AKNS hierarchy, ...

Connections to established concepts

- We can pass between the pluri-Lagrangian and Hamiltonian formalisms for 1-forms* and 2-forms†.

  The Hamiltonians are in involution if and only if $\mathrm{d}\mathcal{L} = 0$ on solutions.
- Lagrangian 2-forms can be derived from matrix Lax pairs with a rational dependence on the spectral parameter.‡
- The flows of a pluri-Lagrangian system are variational symmetries of each other if and only if $\mathrm{d}\mathcal{L} = 0$ on solutions.§

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* Suris. Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geometric Mechanics, 2013

† V. Hamiltonian structures for integrable hierarchies of Lagrangian PDEs Open Communications in Nonlinear Mathematical Physics, 2021.

‡ Sleigh, Nijhoff, Caudrelier. A variational approach to Lax representations. Journal of Geometry and Physics, 2019.

§ Petrera, V. Variational symmetries and pluri-Lagrangian structures for integrable hierarchies of PDEs. European Journal of Mathematics, 2021

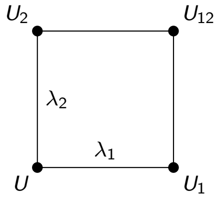
Discretisation of Hamiltonian systems

Hamiltonian ODE	$\rightarrow$	symplectic map
Liouville-Arnold system	$\rightarrow$	commuting symplectic maps (or symplectic map with conserved quantities?)
Hamiltonian PDE	$\rightarrow$	partial difference equation: multisymplectic map on a lattice?

Quad equations

$$\mathcal{Q}(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2) = 0$$

- Subscripts of U denote lattice shifts.
- λ_1, λ_2 are parameters.
- Invariant under symmetries of the square, affine in each of U, U_1, U_2, U_{12}.



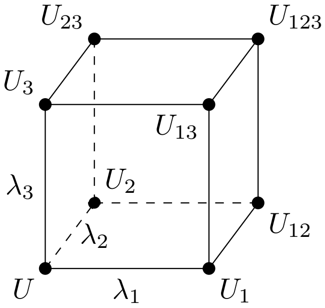
Discrete analogue of commuting flows:

Multi-dimensional consistency

The three ways of calculating U_{123}, using

$$\mathcal{Q}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) = 0,$$

and its shifts, give the same result.



Example: lattice potential KdV:

$$(U - U_{12})(U_1 - U_2) - \lambda_1 + \lambda_2 = 0$$

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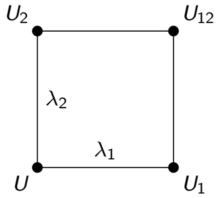

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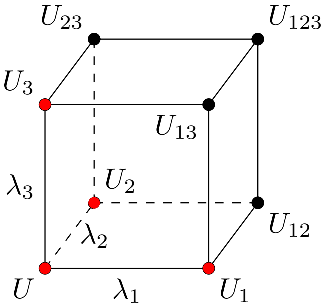

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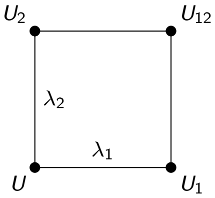

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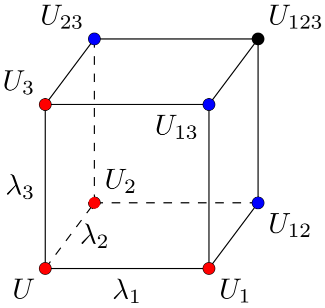

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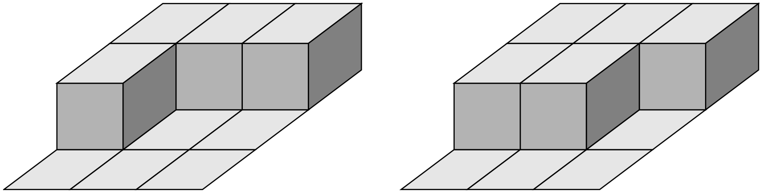
Variational principle for quad equations

For some discrete 2-form

$$\mathcal{L}(\square_{ij}) = \mathcal{L}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j),$$

the action $\sum_{\square \in \Gamma} \mathcal{L}(\square)$ is critical on all 2-surfaces Γ in $\mathbb{Z}^N$ simultaneously.



Discretising Hamiltonian structures is ambiguous. But the discrete and continuous variational principles are essentially the same.

Lobb, Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009.

Summary

- The pluri-Lagrangian (or Lagrangian multiform) principle is a widely applicable characterization of integrability:

  Applies to ODEs and PDEs, discrete and continuous.

- Closedness of the Lagrangian form, i.e. $\mathrm{d}\mathcal{L} = 0$, is related to variational symmetries and Hamiltonians in involution.

- Construction of Lagrangian 1- and 2-forms using:
  - Variational symmetries
  - Hamiltonian structures
  - Continuum limits
  - ...

To do

Work in progress:

- A non-abelian symmetry group can be captured by using a Lie group as multi-time instead of $\mathbb{R}^N$.
- Application to semi-discrete systems.

Further questions:

- Relation to bi-Hamiltonian structures
- Classification of Lagrangian multi-forms.
- Application to infinite-dimensional symmetry groups

  $\hookrightarrow$ Noether's second theorem.
- . . .

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Thank you for listening! **¡Gracias por escuchar!**

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