Loughborough University

Lagrangians in integrable systems

a variational principle for symmetries

Mats Vermeeren

Landau seminar

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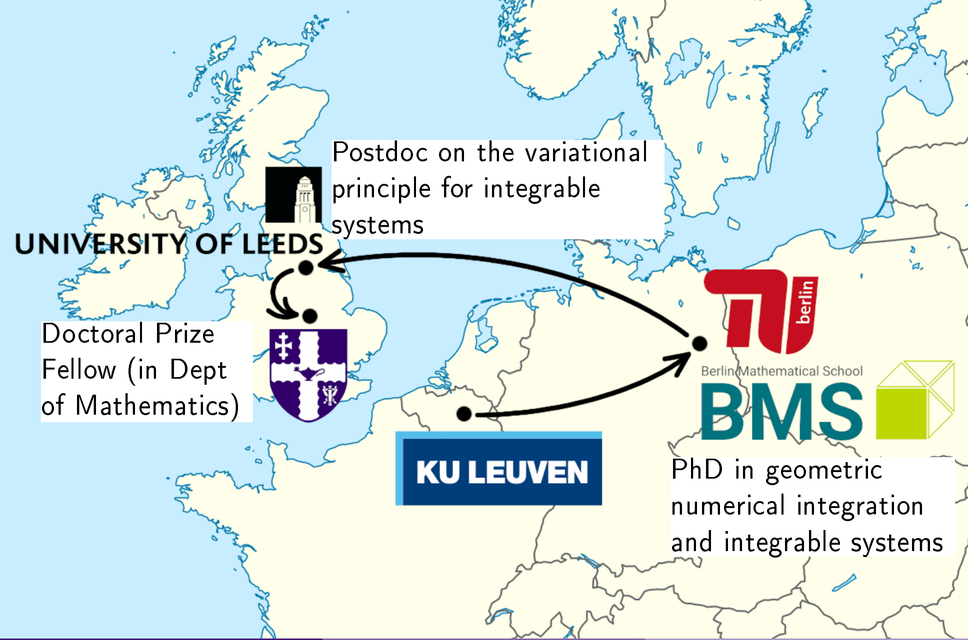
About me


Postdoc on the variational principle for integrable systems
UNIVERSITY OF LEEDS
Doctoral Prize Fellow (in Dept of Mathematics)
KU LEUVEN
berlin
Berlin Mathematical School
BMS
PhD in geometric numerical integration and integrable systems

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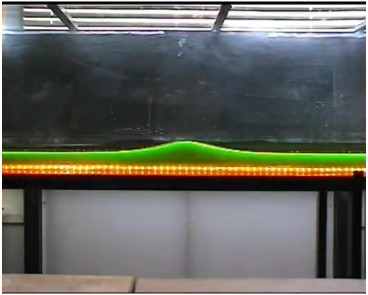
Linear vs nonlinear differential equations

Linear: nice solutions and properties: traveling waves, superposition principle

Nonlinear: often chaotic, difficult to understand

Integrable: nonlinear but "nice"

Example: KdV equation $v_t = v_{xxx} + 6vv_x$ with soliton solutions



https://youtu.be/hfc3IL9gAts

Soliton interaction

Asymptotic behaviour: like superposition, but with phase shift.

Integrable systems

Most nonlinear differential equations (wether ODE or PDE) are impossible to solve explicitly.

Integrable systems are the exception. They have some underlying structure which helps us. Often, this structure consists of a number of symmetries:

An equation is integrable if has sufficiently many symmetries.

Each symmetry, in it infinitesimal form, defines a differential equation. Hence:

An equation is integrable if it is part of a sufficiently large family of compatible equations.

A common interpretation of "compatible" is given in terms of Hamiltonian mechanics.

Classical mechanics

Hamiltonian mechanics

Hamilton function

$$H : \mathbb{R}^{2N} \cong T^*Q \to \mathbb{R} :$$
$$(q,p) \mapsto H(q,p)$$

Typically

$$H(q,p) = \frac{1}{2m} p^2 + U(q)$$

Dynamics given by canonical equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Flow consists of symplectic maps and preserves H.

Lagrangian mechanics

Lagrange function

$$L : \mathbb{R}^{2N} \cong TQ \to \mathbb{R} :$$
$$(q,\dot{q}) \mapsto L(q,\dot{q})$$

Typically

$$L(q,\dot{q}) = \frac{m}{2}\dot{q}^2 - U(q)$$

Dynamics follows curves which are minmizers (critical points) of the action

$$\int L(q,\dot{q}) \, \mathrm{d}t$$

Poisson Brackets

Poisson bracket of two functions on T^*Q:

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$$

Dynamics of a Hamiltonian system:

$$\dot{q}_i = \{H, q_i\}, \qquad \dot{p}_i = \{H, p_i\}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} f(q,p) = \{H, f\}$$

In particular: f is conserved if and only if $\{H, f\} = 0$.

A Hamiltonian system with Hamilton function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \ldots H_N$ such that $\{H_i, H_j\} = 0$.

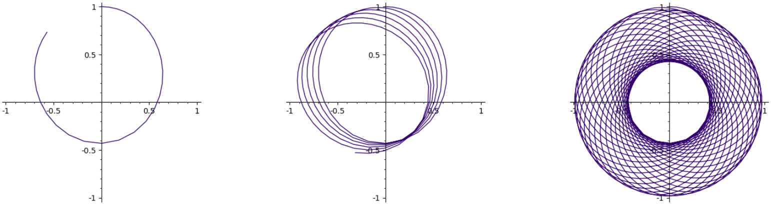
- Each H_i defines its own flow: N dynamical systems.
- Each H_i is a conserved quantity for all flows.

Liouville-Arnold integrability

A Hamiltonian system with Hamilton function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \ldots H_N$ such that $\{H_i, H_j\} = 0$.

- The dynamics is confined to a leaf of the foliation $\{H_i = \text{const}\}$.
- If this foliation is compact, its leaves are tori.

  Example: central force in the plane:



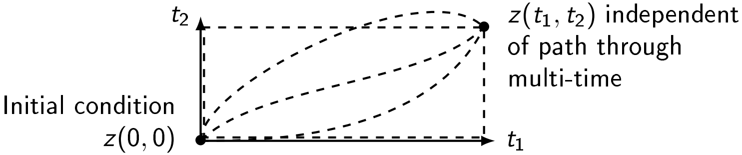
- Dynamics on these tori are linear in action-angle variables.
- The flows commute.

Two commuting flows

Let $z = (q, p)$. Consider two Hamiltonian ODEs

$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_1} = \{f(z), H_1(z)\} \qquad \text{with } \{H_1, H_2\} = 0$$
$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_2} = \{f(z), H_2(z)\}$$

The flows commute, meaning that evolution can be parameterised by the (t_1, t_2) plane, called multi-time.



Additional commuting equations can be accommodated by increasing the dimension of multi-time: $\mathbb{R}^n$ instead of $\mathbb{R}^2$.

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Lagrangian formulation of Liouville integrable system

On the Hamiltonian side, commutativity is implied by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Suppose we have Lagrange functions L_i associated to H_i.

Lagrangian multi-form (Pluri-Lagrangian) principle for ODEs

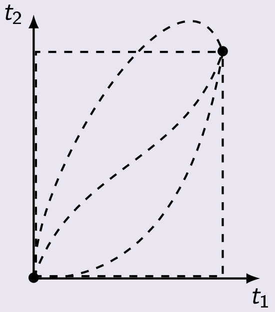
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q] \, \mathrm{d}t_i .$$

Look for dynamical variables $q(t_1, \ldots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$



Multi-time Euler-Lagrange equations

Assume that

$$L_1[q] = L_1(q, q_{t_1}) \qquad \text{and} \qquad L_i[q] = L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1$$

Then the multi-time Euler-Lagrange equations for $\mathcal{L} = \sum_i L_i[q] \, \mathrm{d}t_i$ are

Usual Euler-Lagrange equations: $\dfrac{\partial L_i}{\partial q} - \dfrac{\mathrm{d}}{\mathrm{d}t_i} \dfrac{\partial L_i}{\partial q_{t_i}} = 0$

Usual EL wrt to alien derivatives: $\dfrac{\partial L_i}{\partial q_{t_1}} - \dfrac{\mathrm{d}}{\mathrm{d}t_i} \dfrac{\partial L_i}{\partial q_{t_1 t_i}} = 0, \quad i \neq 1$

Additional conditions: $\dfrac{\partial L_i}{\partial q_{t_i}} = \dfrac{\partial L_j}{\partial q_{t_j}}$

Example: Kepler Problem

The classical Lagrangian

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e \qquad (e \text{ fixed unit vector})$$

into a Lagrangian 1-form $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$.

Multi-time Euler-Lagrange equations:

$$\frac{\partial L_1}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L_1}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_1 t_1} = -\frac{q}{|q|^3} \qquad \text{(Keplerian motion)}$$

$$\frac{\partial L_2}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_2}\frac{\partial L_2}{\partial q_{t_2}} = 0 \quad \Rightarrow \quad q_{t_1 t_2} = e \times q_{t_1}$$

$$\frac{\partial L_2}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_2} = e \times q \qquad \text{(Rotation)}$$

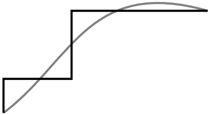
$$\frac{\partial L_1}{\partial q_{t_1}} = \frac{\partial L_2}{\partial q_{t_2}} \quad \Rightarrow \quad q_{t_1} = q_{t_1}$$

Derivation of the multi-time Euler-Lagrange equations

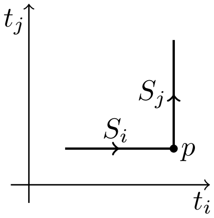
Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[q] \, \mathrm{d}t_i$

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.

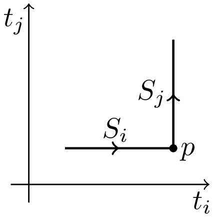


Variations are local, so it is sufficient to look at an L-shaped curve $S = S_i \cup S_j$.



Derivation of the multi-time Euler-Lagrange equations

On one of the straight pieces, S_i $(i \neq 1)$, we get



$$\delta \int_{S_i} L_i \, \mathrm{d}t_i = \int_{S_i} \left(\frac{\partial L_i}{\partial q} \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} + \frac{\partial L_i}{\partial q_{t_i}} \delta q_{t_i} \right) \mathrm{d}t_i$$

Integration by parts (wrt t_i only) yields

$$\delta \int_{S_i} L_i \, \mathrm{d}t_i = \int_{S_i} \left(\left(\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} \right) \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} \right) \mathrm{d}t_i + \left. \frac{\partial L_i}{\partial q_{t_i}} \delta q \right|_p$$

Since p is an interior point of the curve, we cannot set $\delta q(p) = 0$!

Arbitrary δq and δq_{t_1} so we find:

Multi-time Euler-Lagrange equations

$$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_1}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}}$$

Higher order Lagranigans

Multi-time Euler-Lagrange equations

Usual Euler-Lagrange equations: $\dfrac{\delta_i L_i}{\delta q_I} = 0 \qquad \forall I \not\ni t_i,$

Additional conditions: $\dfrac{\delta_i L_i}{\delta q_{It_i}} = \dfrac{\delta_j L_j}{\delta q_{It_j}} \qquad \forall I,$

where

- I is a multi-index, q_I the corresponding partial derivative
- $\dfrac{\delta_i}{\delta q_I}$ is the variational derivative in the direction of t_i:

$$\frac{\delta_i L_i}{\delta q_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial L_i}{\partial q_{It_i^\alpha}} = \frac{\partial L_i}{\partial q_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{It_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_i}{\partial q_{It_i^2}} - \dots$$

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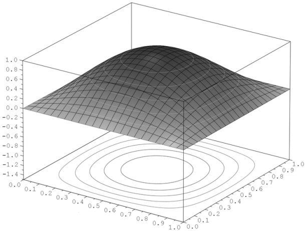
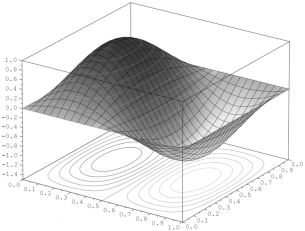
Pluri-Lagrangian principle for PDEs ($d = 2$)

Given a 2-form

$$\mathcal{L}[q] = \sum_{i,j} L_{ij}[q] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $q : \mathbb{R}^N \to \mathbb{R}$, such that $\displaystyle\int_\Gamma \mathcal{L}[q]$ is critical on all smooth 2-surfaces Γ in multi-time $\mathbb{R}^N$, w.r.t. variations of q.



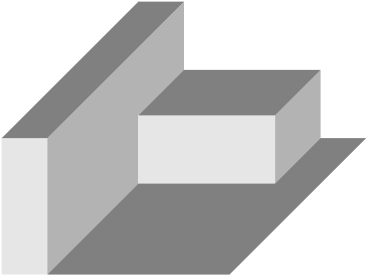


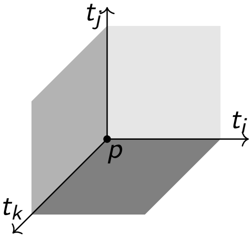
Multi-time EL equations

Consider a Lagrangian 2-form $\mathcal{L}[q] = \sum_{i,j} L_{ij}[q]\,\mathrm{d}t_i \wedge \mathrm{d}t_j$.

Every smooth surface can be approximated arbitrarily well by stepped surfaces.

It is sufficient to require criticality on stepped surfaces. Variations can be taken locally, so it is sufficient to consider elementary corners.





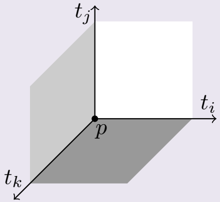
Multi-time EL equations

for $\mathcal{L}[q] = \sum_{i,j} L_{ij}[q]\, \mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{I t_j}} = \frac{\delta_{ik} L_{ik}}{\delta q_{I t_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{I t_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta q_{I t_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta q_{I t_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial q_{I t_i^\alpha t_j^\beta}}$$

Example: NLS equation

The NonLinear Schrödinger equation

$$iq_{t_2} = -\frac{1}{2} q_{xx} + |q|^2 q,$$

is a "universal" weakly nonlinear, dispersive, energy-preserving equation, appearing in

- nonlinear optics
- condensed matter physics
- fluid and plasma dynamics
- ...

The NLS equations has an infinite hierarchy of symmetries (AKNS hierarchy). The first symmetry is

$$q_{t_3} = -\frac{1}{4} q_{xxx} + \frac{3}{2} |q|^2 q_x,$$

where we identify $t_1 = x$.

Example: NLS equation

The equations

$$iq_{t_2} = -\frac{1}{2} q_{xx} + |q|^2 q,$$
$$q_{t_3} = -\frac{1}{4} q_{xxx} + \frac{3}{2} |q|^2 q_x$$

are multi-time EL equations of a 2-form

$$\mathcal{L} = L_{12} \, \mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13} \, \mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23} \, \mathrm{d}t_2 \wedge \mathrm{d}t_3$$

- L_{12}, L_{12}: Lagrange functions for the individual equations:

$$L_{12} = \mathrm{Im}(q^* q_{t_2}) + \frac{1}{2} |q_{t_1}|^2 + \frac{1}{4} |q|^4,$$
$$L_{13} = \mathrm{Im}(q^* q_3) - \frac{1}{4} \mathrm{Im}(q^*_{t_1} q_{t_1 t_1}) - \frac{3}{4} |q|^2 \mathrm{Im}(q^* q_{t_1}).$$

- L_{23}: has no interpretation in classical variational principle.

  The existence of a suitable L_{23} indicates that the t_3-flow is indeed a symmetry of the NLS equation.

Example of a Lagrangian 4-form: Maxwell equations

Consider the electromagnetic 4-potential $A = (A^0, A^1, A^2, A^3)$ the electromagnetic tensor

$$F^{\rho\sigma} = A^{\sigma,\rho} - A^{\rho,\sigma} = \partial^\rho A^\sigma - \partial^\sigma A^\rho$$

Maxwell's equations (in absence of currents)

$$\partial_\rho F^{\rho\sigma} = \partial_\rho \left(\partial^\rho A^\sigma - \partial^\sigma A^\rho \right) = 0$$

Local gauge symmetry: $A^\rho \mapsto A^\rho + \partial^\rho a$ for any function a of spacetime.

Choose a number of such functions a^i.

Gauge symmetry in infinitesimal form: $\partial^i A^\rho = \partial^\rho a^i \quad \Leftrightarrow \quad A^{\rho,i} = a^{i,\rho}$

Goal

Construct a 4-form in space-time variables t_0, t_1, t_2, t_3 (greek indices) and symmetry variables $t_4, t_5, \ldots$ (latin indices) which produces these equations.

Example of a Lagrangian 4-form: Maxwell equations

- Space-time variables t_0, t_1, t_2, t_3 (greek indices)
- Symmetry variables $t_4, t_5, \ldots$ (latin indices)
- $(\pi, \rho, \sigma, \tau)$ even permuation of $(0, 1, 2, 3)$

Coefficients of 4-form $\mathcal{L}$ given by

$$\begin{aligned}
L_{\pi\rho\sigma\tau} = L_{0123} &= \sum_{\mu,\nu} \left(\frac{1}{2} A^{\mu,\nu} A_{\mu,\nu} - \frac{1}{2} A^{\mu,\nu} A_{\nu,\mu} \right), \\
L_{\pi\rho\sigma i} &= \sum_{\mu} \left(A^{\mu,\tau} - A^{\tau,\mu} \right) \left(A_{\mu,i} - a_{i,\mu} \right), \\
L_{\pi\rho j i} &= \left(a_{j,i} - a_{i,j} \right) \left(A^{\sigma,\tau} - A^{\tau,\sigma} \right), \\
L_{\pi k j i} &= 0 \qquad L_{\ell k j i} = 0
\end{aligned}$$

Can take Euler-Lagrange equations wrt A^ρ but also wrt gauge function a^i.

There are infinitely many possible gauge functions a

Relation to Noether's second theorem?

Contents

Exterior derivative of $\mathcal{L}$

Revisit the Kepler problem: $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$ with

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e \qquad (e \text{ fixed unit vector})$$

Multi-time Euler-Lagrange equations:

$$q_{t_1 t_1} = -\frac{q}{|q|^3}$$

$$q_{t_2} = e \times q$$

Coefficient of $\mathrm{d}\mathcal{L}$

$$\frac{\mathrm{d}L_2}{\mathrm{d}t_1} - \frac{\mathrm{d}L_1}{\mathrm{d}t_2} = \left(q_{t_1 t_1} + \frac{q}{|q|^3} \right) (q_{t_2} - e \times q)$$

General observation (also for PDEs): $\mathrm{d}\mathcal{L}$ has a "double zero" on solutions.

Hamiltonian formulation and $\mathrm{d}\mathcal{L}$

We can pass between the pluri-Lagrangian and Hamiltonian formalisms for 1-form and 2-forms.

Lemma ($\mathrm{d}\mathcal{L}$ for 1-forms)

On solutions there holds $\dfrac{\mathrm{d}L_j}{\mathrm{d}t_i} - \dfrac{\mathrm{d}L_i}{\mathrm{d}t_j} = \{H_j, H_i\}$.

It follows that:

Theorem

The Hamiltonians are in involution if and only if $\mathrm{d}\mathcal{L} = 0$ on solutions.

A similar result holds for 2-forms (and presumably for higher forms)

Variational Symmetries and $\mathrm{d}\mathcal{L}$

Connection provided by the closedness property $\mathrm{d}\mathcal{L} = 0$:

1-forms: If $\mathrm{d}\left(\sum_i L_i \, \mathrm{d}t_i \right) = 0$, then $\dfrac{\mathrm{d}L_k}{\mathrm{d}t_j} = \dfrac{\mathrm{d}L_j}{\mathrm{d}t_k}$

$\Rightarrow$ t_j-flow changes L_k by a t_k-derivative.

$$\Rightarrow \partial_j \int_a^b \mathrm{d}L_k \, \mathrm{d}t_k = [L_j]_a^b = \mathrm{const}$$

$\Rightarrow$ flows are variational symmetries of each other.

A similar result holds for higher forms.

Idea: use variational symmetries to construct Lagrangian form.

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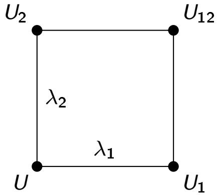
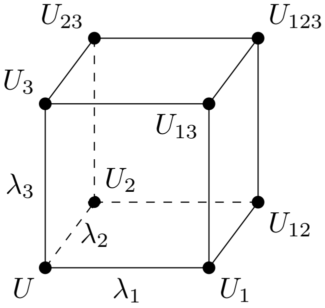
Quad equations

$$\mathcal{Q}(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2) = 0$$



- Subscripts of U denote lattice shifts.
- λ_1, λ_2 are parameters.
- Invariant under symmetries of the square, affine in each of U, U_1, U_2, U_{12}.

Discrete analogue of commuting flows:



Multi-dimensional consistency

The three ways of calculating U_{123}, using

$$\mathcal{Q}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) = 0,$$

and its shifts, give the same result.

Example: lattice potential KdV:

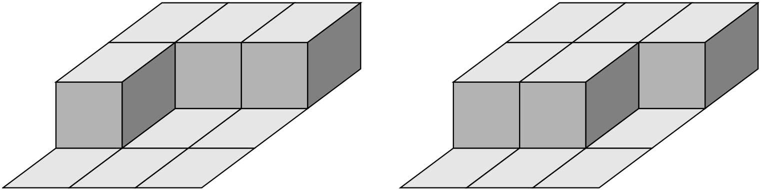
$$(U - U_{12})(U_1 - U_2) - \lambda_1 + \lambda_2 = 0$$

Variational principle for quad equations

For some discrete 2-form

$$\mathcal{L}(\square_{ij}) = \mathcal{L}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j),$$

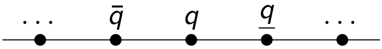
the action $\displaystyle\sum_{\square \in \Gamma} \mathcal{L}(\square)$ is critical on all 2-surfaces Γ in $\mathbb{Z}^N$ simultaneously.



The discrete and continuous variational principles are essentially the same.

Semi-discrete systems

Consider particles on a line: 1 discrete dimension, many continuous times



Toda lattice: exponential nearest-neighbour interaction

$$q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q}).$$

Part of a hierarchy. First symmetry:

$$q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q})$$

(Subscripts stand for partial derivatives: $q_1 = q - t_1$ etc.)

Toda lattice

Lagrangians ("0" for discrete direction)

$$L_{01} = \frac{1}{2}q_1^2 - \exp(\bar{q} - q)$$
$$L_{02} = q_1 q_2 - \frac{1}{3}q_1^3 - (q_1 + \bar{q}_1)\exp(\bar{q} - q)$$
$$L_{12} = -\frac{1}{4}\left(q_2 - q_{11} - q_1^2\right)^2$$

Euler-Lagrange equations:

$$\frac{\delta_{01} L_{01}}{\delta q} = 0 \qquad \to \qquad q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q})$$
$$\frac{\delta_{02} L_{02}}{\delta q_1} = 0 \qquad \to \qquad q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q})$$
$$\frac{\delta_{12} L_{12}}{\delta q} = 0 \qquad \to \qquad \frac{1}{2}q_{22} - q_{11}q_2 - 2q_{12}q_1 - \frac{1}{2}q_{1111} + 3q_1^2 q_{11} = 0$$

Lagrangian formalism produces a non-trivial PDE at a single lattice site.

Contents

Summary

- The pluri-Lagrangian (or Lagrangian multiform) principle describes symmetries and integrability.

  Applies to ODEs and PDEs, discrete and continuous.
- Closedness of the Lagrangian form, i.e. $\mathrm{d}\mathcal{L} = 0$, is related to variational symmetries (Noether) and Hamiltonians in involution.
- Construction of Lagrangian 1- and 2-forms can be done using:
  - Variational symmetries
  - Hamiltonian structures
  - Continuum limits, Lax pairs, ...
- Some open questions:
  - Full development for semi-discrete systems
  - Better understanding of application to gauge theory (∞-dim symmetry groups $\rightarrow$ Noether's second theorem)
  - Application to quantum integrable systems, path integrals, ...

Selected references

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Thank you for your attention!