Loughborough University

Lagrangian multiforms and the Toda hierarchy

Mats Vermeeren

New Trends in Lagrangian and Hamiltonian Aspects of Integrable Systems

Leeds, 13 May, 2022

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Liouville-Arnold integrability

A Hamiltonian system with Hamilton function $H : T^*Q \cong \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \ldots H_N$ such that $\{H_i, H_j\} = 0$.

- Each H_i defines its own flow: N dynamical systems
- Each H_i is a conserved quantity for all flows.
- The dynamics is confined to a leaf of the foliation $\{H_i = \text{const}\}$.
- If this foliation is compact, its leaves are tori.
- Dynamics on these tori are linear in action-angle variables.
- The flows commute:

$$\phi^t_{H_i} \circ \phi^s_{H_j} = \phi^s_{H_j} \circ \phi^t_{H_i}.$$

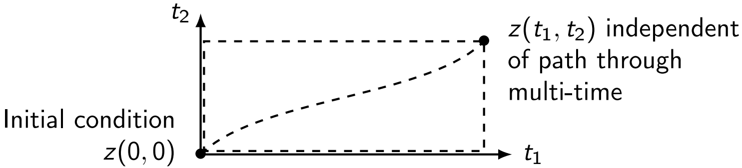
(Infinitesimally: $[X_{H_i}, X_{H_j}] = 0$.)

Two commuting flows

Let $z = (q, p)$. Consider two Hamiltonian ODEs

$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_1} = \{f(z), H_1(z)\} \qquad \text{with } \{H_1, H_2\} = 0$$
$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_2} = \{f(z), H_2(z)\}$$

The flows commute, meaning that evolution can be parameterised by the (t_1, t_2) plane, called multi-time.



Additional commuting equations can be accommodated by increasing the dimension of multi-time: $\mathbb{R}^N$ instead of $\mathbb{R}^2$.

Lagrangian mechanics

Lagrange function $L : TQ \cong \mathbb{R}^{2N} \to \mathbb{R} : (q, q_t) \mapsto L(q, q_t)$

Dynamics follows curves which are minimizers (critical points) of the action

$$\int_a^b L(q, q_t)\,\mathrm{d}t \qquad \text{with fixed boundary values } q(a) \text{ and } q(b).$$

Minimizers satisfy the Euler-Lagrange (EL) equation $\quad \dfrac{\partial L}{\partial q} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial q_t} = 0$

Proof. Consider an arbitrary variation δq:

$$\delta \int_a^b L\,\mathrm{d}t = \int_a^b \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial q_t}\delta q_t \right) \mathrm{d}t$$

Integration by parts yields

$$\delta \int_a^b L\,\mathrm{d}t = \int_a^b \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q_t} \right) \delta q\,\mathrm{d}t + \left[\frac{\partial L}{\partial q_t}\delta q \right]_a^b$$

EL follows because $\delta q(a) = \delta q(b) = 0$ and δq is arbitrary inside (a, b). ■

Overview of Lagrangian multiforms

Includes work by Nijhoff, Suris, Petrera, Sleigh, ...

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Lagrangian formulation of Liouville integrable system

On the Hamiltonian side, commutativity is implied by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Suppose we have Lagrange functions L_i associated to H_i.

Variational ("Pluri-Lagrangian") principle for ODEs

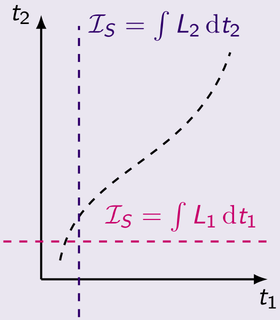
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q] \, \mathrm{d}t_i.$$

Look for dynamical variables $q(t_1, \ldots, t_N)$ such that the action

$$\mathcal{I}_S = \int_S \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve S in multi-time $\mathbb{R}^N$



Multi-time Euler-Lagrange equations

Assume that

$$L_1[q] = L_1(q, q_{t_1}),$$
$$L_i[q] = L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1$$

The multi-time Euler-Lagrange equations for $\mathcal{L} = \sum_i L_i[q] \, \mathrm{d}t_i$ are

Usual Euler-Lagrange equations: $\dfrac{\partial L_i}{\partial q} - \dfrac{\mathrm{d}}{\mathrm{d}t_i} \dfrac{\partial L_i}{\partial q_{t_i}} = 0$

Usual EL wrt to alien derivatives: $\dfrac{\partial L_i}{\partial q_{t_1}} - \dfrac{\mathrm{d}}{\mathrm{d}t_i} \dfrac{\partial L_i}{\partial q_{t_1 t_i}} = 0, \quad i \neq 1$

Additional conditions: $\dfrac{\partial L_i}{\partial q_{t_i}} = \dfrac{\partial L_j}{\partial q_{t_j}}$

Example: Kepler Problem

The classical Lagrangian

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e \qquad (e \text{ fixed unit vector})$$

into a Lagrangian 1-form $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$.

Multi-time Euler-Lagrange equations:

$$\frac{\partial L_1}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L_1}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_1 t_1} = -\frac{q}{|q|^3} \qquad \text{(Keplerian motion)}$$

$$\frac{\partial L_2}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_2}\frac{\partial L_2}{\partial q_{t_2}} = 0 \quad \Rightarrow \quad q_{t_1 t_2} = e \times q_{t_1}$$

$$\frac{\partial L_2}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_2} = e \times q \qquad \text{(Rotation)}$$

$$\frac{\partial L_1}{\partial q_{t_1}} = \frac{\partial L_2}{\partial q_{t_2}} \quad \Rightarrow \quad q_{t_1} = q_{t_1}$$

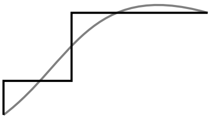
Derivation of the multi-time Euler-Lagrange equations

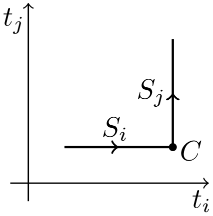
Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[q]\,\mathrm{d}t_i$, with

$$L_1[q] = L_1(q, q_{t_1}),$$
$$L_i[q] = L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1$$

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.



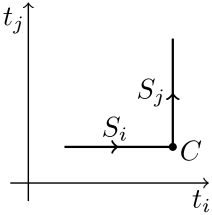


Variations are local, so it is sufficient to look at an L-shaped curve $S = S_i \cup S_j$.

Derivation of the multi-time Euler-Lagrange equations

On one of the straight pieces, S_i ($i \neq 1$), we get

$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \left(\frac{\partial L_i}{\partial q} \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} + \frac{\partial L_i}{\partial q_{t_i}} \delta q_{t_i} \right) \mathrm{d}t_i$$



Integration by parts (wrt t_i only) yields

$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \left(\left(\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} \right) \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} \right) \mathrm{d}t_i + \frac{\partial L_i}{\partial q_{t_i}} \delta q \bigg|_C$$

Since p is an interior point of the curve, we cannot set $\delta q(C) = 0$!

Arbitrary δq and δq_{t_1}, so we find:

Multi-time Euler-Lagrange equations

$$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_1}} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_1 t_i}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}}$$

Higher order Lagranigans $L_i[q] = L_i(q, q_{t_i}, q_{t_i t_j}, \dots)$

For a string $I = t_{i_1} \dots t_{i_k}$ of time variables, denote the corresponding derivative by q_I.

If I is empty then $q_I = q$.

Denote by $\frac{\delta_i}{\delta q_I}$ the variational derivative in the direction of t_i wrt q_I:

$$\begin{aligned} \frac{\delta_i L_i}{\delta q_I} &= \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d} t_i^\alpha} \frac{\partial L_i}{\partial q_{I t_i^\alpha}} \\ &= \frac{\partial L_i}{\partial q_I} - \frac{\mathrm{d}}{\mathrm{d} t_i} \frac{\partial L_i}{\partial q_{I t_i}} + \frac{\mathrm{d}^2}{\mathrm{d} t_i^2} \frac{\partial L_i}{\partial q_{I t_i^2}} - \dots \end{aligned}$$

Multi-time Euler-Lagrange equations

Usual Euler-Lagrange equations: $\displaystyle \frac{\delta_i L_i}{\delta q_I} = 0 \qquad \forall I \not\ni t_i,$

Additional conditions: $\displaystyle \frac{\delta_i L_i}{\delta q_{I t_i}} = \frac{\delta_j L_j}{\delta q_{I t_j}} \qquad \forall I,$

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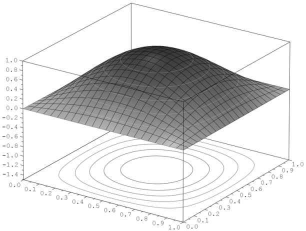
Variational principle for PDEs ($d = 2$)

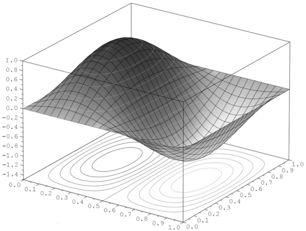
Pluri-Lagrangian principle

Given a 2-form

$$\mathcal{L}[q] = \sum_{i,j} L_{ij}[q]\, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $q : \mathbb{R}^N \to \mathbb{R}$, such that $\displaystyle\int_S \mathcal{L}[q]$ is critical on all smooth surfaces S in multi-time $\mathbb{R}^N$, w.r.t. variations of q.





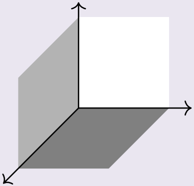
Multi-time EL equations

for $\mathcal{L}[q] = \sum_{i,j} L_{ij}[q] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta q_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta q_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta q_{It_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial q_{I t_i^\alpha t_j^\beta}}$$

Example: Potential KdV hierarchy

$u_{t_2} = Q_2 = u_{xxx} + 3u_x^2,$
$u_{t_3} = Q_3 = u_{xxxxx} + 10u_x u_{xxx} + 5u_{xx}^2 + 10u_x^3,$
where we identify $t_1 = x$.

The differentiated equations $u_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x} Q_i$ are Lagrangian with

$$L_{12} = \frac{1}{2} u_x u_{t_2} - \frac{1}{2} u_x u_{xxx} - u_x^3,$$
$$L_{13} = \frac{1}{2} u_x u_{t_3} - \frac{1}{2} u_{xxx}^2 + 5u_x u_{xx}^2 - \frac{5}{2} u_x^4.$$

A suitable coefficient L_{23} of

$$\mathcal{L} = L_{12}\, \mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}\, \mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}\, \mathrm{d}t_2 \wedge \mathrm{d}t_3$$

can be found (nontrivial task!) in the form

$$L_{23} = \frac{1}{2}(u_{t_3} Q_2 - u_{t_2} Q_3) + p_{23}.$$

Example: Potential KdV hierarchy

- The equations $\dfrac{\delta_{12} L_{12}}{\delta u} = 0$ and $\dfrac{\delta_{13} L_{13}}{\delta u} = 0$ yield
$$u_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x} Q_2 \qquad \text{and} \qquad u_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x} Q_3.$$
- The equations $\dfrac{\delta_{12} L_{12}}{\delta u_x} = \dfrac{\delta_{32} L_{32}}{\delta u_{t_3}}$ and $\dfrac{\delta_{13} L_{13}}{\delta u_x} = \dfrac{\delta_{23} L_{23}}{\delta u_{t_2}}$ yield
$$u_{t_2} = Q_2 \qquad \text{and} \qquad u_{t_3} = Q_3,$$
the evolutionary equations!
- All other multi-time EL equations are corollaries of these.

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Exterior derivative of $\mathcal{L}$

Revisit the Kepler problem: $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$ with

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e \qquad (e \text{ fixed unit vector})$$

Multi-time Euler-Lagrange equations:

$$q_{t_1 t_1} = -\frac{q}{|q|^3}$$

$$q_{t_2} = e \times q$$

Coefficient of $\mathrm{d}\mathcal{L}$

$$\frac{\mathrm{d}L_2}{\mathrm{d}t_1} - \frac{\mathrm{d}L_1}{\mathrm{d}t_2} = \left(q_{t_1 t_1} + \frac{q}{|q|^3} \right) (q_{t_2} - e \times q)$$

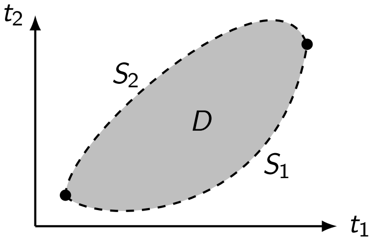
Observation (also for PDEs): $\mathrm{d}\mathcal{L}$ often has a "double zero" on solutions.

$\mathrm{d}\mathcal{L} = 0$ sets a Lagrangian multiform apart from a pluri-Lagrangian system.

Interpretation of closedness condition

If $\mathrm{d}\mathcal{L} = 0$, then the action is invariant wrt variations in geometry

Deforming the curve (surface) of integration leaves action invariant.



$$\int_{S_1} \mathcal{L} - \int_{S_2} \mathcal{L} = \int_D \mathrm{d}\mathcal{L} = 0$$

Recall: before we only took variations of q.

$\mathrm{d}\mathcal{L}$ provides an alternative derivation of the EL equations:

WLOG, we can restrict the variational principle to simple closed curves (surfaces) of integration, i.e. boundaries of submanifolds S. Then

$$\delta \int_{\partial S} \mathcal{L} = \int_S \delta \mathrm{d}\mathcal{L},$$

hence the variational principle is equivalent to $\delta \mathrm{d}\mathcal{L} = 0$.

Multi-time EL equations can be obtained by taking variations of (coefficients of) $\mathrm{d}\mathcal{L}$.

Hamiltonian formulation and $\mathrm{d}\mathcal{L}$

We can pass between the pluri-Lagrangian and Hamiltonian formalisms for 1-form and 2-forms.

Lemma ($\mathrm{d}\mathcal{L}$ for 1-forms)

On solutions there holds $\dfrac{\mathrm{d}L_j}{\mathrm{d}t_i} - \dfrac{\mathrm{d}L_i}{\mathrm{d}t_j} = \{H_j, H_i\}$.

It follows that:

Theorem

The Hamiltonians are in involution if and only if $\mathrm{d}\mathcal{L} = 0$ on solutions.

A similar result holds for 2-forms (and presumably for higher forms)

Variational Symmetries and $\mathrm{d}\mathcal{L}$

$\mathrm{d}\mathcal{L} = 0$ expresses that flows are variational symmetries of each other

$$\mathrm{d}\left(\sum_i L_i \,\mathrm{d}t_i \right) = 0 \Rightarrow \frac{\mathrm{d}L_k}{\mathrm{d}t_j} = \frac{\mathrm{d}L_j}{\mathrm{d}t_k}$$

$\Rightarrow$ t_j-flow changes L_k by a t_k-derivative

$$\Rightarrow \partial_j \int_a^b L_k \,\mathrm{d}t_k = \int_a^b \frac{\mathrm{d}L_j}{\mathrm{d}t_k} \,\mathrm{d}t_k = [L_j]_a^b = \mathrm{const}$$

Adding a constant to the action does not change the dynamics, hence ∂_j is a variational symmetry.

A similar result holds for higher forms.

We can use variational symmetries to construct Lagrangian multiforms.

Semi-discrete Lagrangian 2-forms and the Toda lattice

Joint work with Duncan Sleigh, `arXiv:2204.130633`

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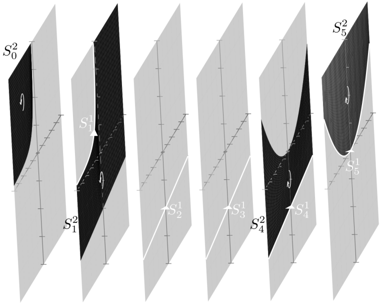
Semi-discrete geometry

We consider only 1 discrete direction: multi-time is $\mathbb{Z} \times \mathbb{R}^N$

A semi-discrete surface is a collection of surfaces and curves in $\mathbb{R}^N$, each assigned a value of $\mathbb{Z}$

Curves (white) are where the surface (black) jumps to a different value of $\mathbb{Z}$



d-dimensional semi-discrete submanifold S

$$S = \left(\bigsqcup_{k \in \mathbb{Z}} S_k^{d-1} \,,\, \bigsqcup_{k \in \mathbb{Z}} S_k^d \right)$$

S_k^{d-1}: disjoint union of oriented $(d-1)$-submanifolds of $\mathbb{R}^N$
S_k^d: disjoint union of oriented d-submanifolds of $\mathbb{R}^N$.

$d = 2$: semi-discrete surface $\qquad$ $d = 3$: semi-discrete volume

Semi-discrete geometry

Boundary of $S = \left(\bigsqcup_{k \in \mathbb{Z}} S_k^{d-1} \, , \, \bigsqcup_{k \in \mathbb{Z}} S_k^d \right)$

$$\partial S = \left(\bigsqcup_{k \in \mathbb{Z}} -\partial S_k^{d-1} \, , \, \bigsqcup_{k \in \mathbb{Z}} \left(\partial S_k^d \sqcup S_k^{d-1} \sqcup -S_{k+1}^{d-1} \right) \right),$$

where the minus sign denotes a change of orientation.

Sign conventions are chosen so that the boundary of a boundary is empty.

Dynamical variables will be (scalar) functions q of $\mathbb{Z} \times \mathbb{R}^N$.

Superscript to emphasise lattice position:

$$q^{[k]} = q(k, t_1, \ldots, t_N)$$

$\mathcal{T}$ denotes shift operator:

$$\mathcal{T} q^{[k]} = q^{[k+1]}$$

Semi-discrete geometry

- semi-discrete d-form
$$\mathcal{L}[q] = \Big(\mathcal{L}^{d-1}[q] \, , \, \mathcal{L}^{d}[q] \Big)$$
consists of a $(d-1)$-form and a d-form, with coefficients depending on phase space variables.
- The semi-discrete integral over semi-discrete submanifold S
$$\int_S \mathcal{L}[q] = \sum_k \int_{S_k^{d-1}} \mathcal{L}^{d-1}\Big[q^{[k]}\Big] + \sum_k \int_{S_k^{d}} \mathcal{L}^{d}\Big[q^{[k]}\Big] \, ,$$
- The exterior derivative:
$$\mathbb{d}\mathcal{L} = \Big(\Delta(\mathcal{L}^d) - \mathrm{d}\mathcal{L}^{d-1} \, , \, \mathrm{d}\mathcal{L}^{d} \Big) \, ,$$
where $\Delta = \mathrm{id} - \mathcal{T}^{-1}$ is the backward difference operator.
- Stokes theorem:
$$\int_S \mathbb{d}\mathcal{L} = \int_{\partial S} \mathcal{L}.$$

Variational principle in semi-discrete multi-time

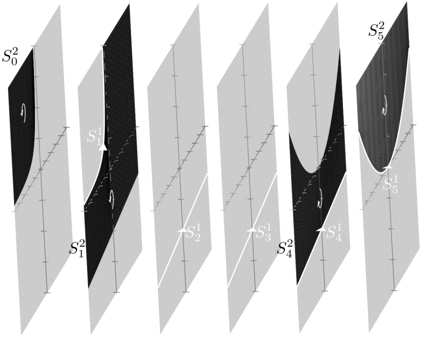
Consider a semi-discrete 2-form

$$\left(\sum_j L_{0j} \,\mathrm{d}t_j \;,\; \sum_{i,j} L_{ij} \,\mathrm{d}t_i \wedge \mathrm{d}t_j \right)$$

Look for dynamical variables $q(k, t_1, \ldots, t_N)$ such that the action

$$\mathcal{I}_S = \int_S \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every semi-discrete surface S in multi-time $\mathbb{R}^N$



Semi-discrete variational derivatives

$$\frac{\delta_0 L}{\delta q_I} := \frac{\partial}{\partial q_I} \sum_{n \in \mathbb{N}} \mathcal{T}^{-n} L,$$

$$\frac{\delta_{0i} L}{\delta q_I} := \frac{\delta_i}{\delta q_I} \sum_{n \in \mathbb{N}} \mathcal{T}^{-n} L.$$

Traditional discrete EL eqn:

$$\sum_{n \in \mathbb{Z}} \mathcal{T}^{-n} \frac{\partial L}{\partial q^{[n]}} = 0$$

Traditional semi-discrete EL eqn:

$$\sum_{n \in \mathbb{Z}} \mathcal{T}^{-n} \frac{\delta_i L}{\delta q^{[n]}} = 0$$

Same if L does not contain negative shifts of q.

Denote $\bar{q} = \mathcal{T} q$ and $\underline{q} = \mathcal{T}^{-1} q$.

Examples:

$$\frac{\delta_{0i} q_{t_i}^2}{\delta q} = \frac{\delta_i q_{t_i}^2}{\delta q} = -2 \frac{\mathrm{d}}{\mathrm{d} t_i} q_{t_i} = -2 q_{t_i t_i},$$

$$\frac{\delta_{0i} \bar{q}_{t_i}^2}{\delta q} = -2 \frac{\mathrm{d}}{\mathrm{d} t_i} q_{t_i} = -2 q_{t_i t_i},$$

$$\frac{\delta_{0i} q \bar{q}}{\delta q} = \frac{\delta_0 q \bar{q}}{\delta q} = \bar{q} + \underline{q},$$

$$\frac{\delta_{0i} q \underline{q}}{\delta q} = \frac{\delta_0 q \underline{q}}{\delta q} = \underline{q}.$$

Semi-discrete multi-time Euler-Lagrange equations

A field is critical if and only if the following multi-time Euler-Lagrange equations hold for all $n \in \mathbb{Z}$:

$$\frac{\delta_{ij} L_{ij}}{\delta q_I^{[n]}} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_j}^{[n]}} - \frac{\delta_{ik} L_{ik}}{\delta q_{It_k}^{[n]}} = 0 \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_i t_j}^{[n]}} + \frac{\delta_{jk} L_{jk}}{\delta q_{It_j t_k}^{[n]}} + \frac{\delta_{ki} L_{ki}}{\delta q_{It_k t_i}^{[n]}} = 0 \qquad \forall I,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_j}^{[n]}} + \frac{\delta_{0i} L_{0i}}{\delta q_I^{[n]}} = 0 \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_i t_j}^{[n]}} - \frac{\delta_{0j} L_{0j}}{\delta q_{It_j}^{[n]}} + \frac{\delta_{0i} L_{0i}}{\delta q_{It_i}^{[n]}} = 0 \qquad \forall I,$$

If n is such that L_{ij} does not depend on $q_I^{[n]}$ for any I, then

$$\frac{\delta_{0i} L_{0i}}{\delta q_I^{[n]}} = 0 \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{0j} L_{0j}}{\delta q_{It_j}^{[n]}} - \frac{\delta_{0i} L_{0i}}{\delta q_{It_i}^{[n]}} = 0 \qquad \forall I.$$

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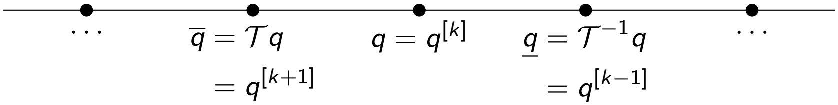
Toda Lattice

Consider particles on a line: 1 discrete dimension, many continuous times



Denote $\quad q_1 = q_{t_1} = \frac{\mathrm{d}q}{\mathrm{d}t_1}, \quad q_{11} = q_{t_1 t_1} = \frac{\mathrm{d}^2 q}{\mathrm{d}t_1^2}, \quad$ etc.

Toda lattice: exponential nearest-neighbour interaction

$$q_{11} = \exp(\overline{q} - q) - \exp(q - \underline{q}).$$

It is part of a hierarchy:

$$\begin{aligned} q_2 &= q_1^2 + \exp(\overline{q} - q) + \exp(q - \underline{q}) \\ q_3 &= q_1^3 + (2q_1 + \underline{q}_1) \exp(q - \underline{q}) + (2q_1 + \overline{q}_1) \exp(\overline{q} - q) \\ &\dots \end{aligned}$$

Toda lattice

Each member of the Toda hierarchy is Hamiltonian with Hamilton function of the form

$$H_i = \sum_{\alpha \in \mathbb{Z}} \mathcal{T}^\alpha h_i = \ldots + \underline{h_i} + h_i + \overline{h_i} + \ldots$$

Define $L_{0j} = q_1 q_j - h_j$ and L_{ij} in such a way that $\mathrm{d}\mathcal{L}$ will have a double zero on solutions of the hierarchy.

This is possible because the H_i are in involution.
We have a formula for L_{ij}, which this slide is too small to contain.

Semi-discrete Lagrangian 2-form $\left(\sum_j L_{0j} \, \mathrm{d}t_j \,,\ \sum_{i,j} L_{ij} \, \mathrm{d}t_i \wedge \mathrm{d}t_j \right)$ with

$$L_{01} = \frac{1}{2} q_1^2 - \exp(\bar{q} - q)$$

$$L_{02} = q_1 q_2 - \frac{1}{3} q_1^3 - (q_1 + \bar{q}_1) \exp(\bar{q} - q)$$

$$L_{12} = -\frac{1}{4} \left(q_2 - q_{11} - q_1^2 \right)^2$$

Toda lattice

Euler-Lagrange equations:

$$\frac{\delta_{01} L_{01}}{\delta q} = 0 \quad \rightarrow \quad q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q})$$

$$\frac{\delta_{02} L_{02}}{\delta q_1} = 0 \quad \rightarrow \quad q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q})$$

$$\frac{\delta_{12} L_{12}}{\delta q} = 0 \quad \rightarrow \quad \frac{1}{2} q_{22} - q_{11} q_2 - 2 q_{12} q_1 - \frac{1}{2} q_{1111} + 3 q_1^2 q_{11} = 0 \qquad (*)$$

By construction, the variational principle is satisfied by the Toda hierarchy, so $(*)$ must be a consequence of the differential-difference equations

Is $(*)$ itself an integrable PDE?

The first two Toda equations can we written as

$$\exp(\bar{q} - q) = \frac{1}{2}(q_2 + q_{11} - q_1^2), \qquad \exp(q - \underline{q}) = \frac{1}{2}(q_2 - q_{11} - q_1^2).$$

Auto-Bäcklund transformation for $(*)$?

Toda lattice

A the next level we find

$$
\begin{aligned}
L_{03} &= q_1 q_3 - \frac{1}{4} q_1^4 - a(q_1^2 + \overline{q}_1^2 + q_1 \overline{q}_1) - a\overline{a} + \frac{1}{2} a^2, \\
L_{13} &= -a \left(\overline{q}_1^3 + 2a\overline{q}_1 + \overline{a}\overline{\overline{q}}_1 + 2\overline{q}_1 \overline{q}_{11} + q_1 \overline{q}_{11} - \overline{q}_3 + a q_1 - \overline{a} q_1 \right), \\
L_{23} &= -a \Big(\overline{q}_2 \left(q_1^2 + \overline{q}_1^2 + q_1 \overline{q}_1 + \underline{a} + a \right) + \overline{\overline{q}}_2 \overline{a} + 2\overline{q}_1 \overline{q}_{12} + q_1 \overline{q}_{12} - \overline{q}_{13} \\
&\qquad - \overline{q}_3 \left(q_1 + \overline{q}_1 \right) - q_1^2 \overline{q}_1^2 - \underline{a}\overline{q}_1^2 + 2a q_1 \overline{q}_1 - \overline{a} q_1^2 - a\overline{a} - \underline{a}\overline{a} - \underline{a}a - a^2 \Big)
\end{aligned}
$$

where $a = \exp(\bar{q} - q)$

Again we can use the multi-time Euler-Lagrange equations to obtain a PDE at a single lattice site:

$$
\frac{\delta_{13} L_{13}}{\delta q} = 0 \quad \rightarrow \quad q_1^3 - 3q_1 q_{11} + 6q_1 a + q_{111} - q_3 = 0,
$$

which can be simplified to

$$
q_3 = -2q_1^3 + 3q_1 q_2 + q_{111}.
$$

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Summary

- The Lagrangian multiform (or pluri-Lagrangian) principle describes symmetries and integrability.

  Applies to ODEs and PDEs, discrete and continuous.

- Closedness of the Lagrangian form, i.e. $\mathrm{d}\mathcal{L} = 0$, is related to variational symmetries and Hamiltonians in involution.

  $\delta \mathrm{d}\mathcal{L} = 0$ is equivalent to the variational problem.

- We constructed semi-discrete Lagrangian 2-from for the Toda hierarchy. It reveals that integrable (?) PDEs are hidden within.

References

Main
- Sleigh, V. Semi-discrete Lagrangian 2-forms and the Toda hierarchy `arXiv:2204.130633`

Background
- Lobb, Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009.
- Yoo-Kong, Lobb, Nijhoff. Discrete-time Calogero–Moser system and Lagrangian 1-form structure. J Phys. A. 2011.
- Suris. Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geometric Mechanics, 2013
- Suris, V. On the Lagrangian structure of integrable hierarchies. In: Advances in Discrete Differential Geometry, Springer. 2016.
- Petrera, Suris. Variational symmetries and pluri-Lagrangian systems in classical mechanics. J. Nonlin. Math. Phys., 2017.
- V. Hamiltonian structures for integrable hierarchies of Lagrangian PDEs Open Communications in Nonlinear Mathematical Physics, 2021.

Thank you for your attention!