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A variational principle for integrable systems

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Linear vs nonlinear differential equations

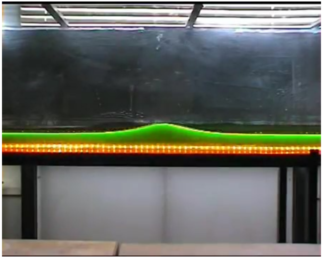
Linear: nice solutions and properties:

- travelling waves
- superposition principle

Nonlinear: often chaotic, difficult to understand

Integrable: nonlinear but "nice"

Example: KdV equation $v_t = v_{xxx} + 6vv_x$ with soliton solutions



https://youtu.be/hfc3IL9gAts

Soliton interaction

Asymptotic behaviour: like superposition, but with phase shift.

Integrable systems

Most nonlinear differential equations (whether ODE or PDE) are impossible to solve explicitly.

Integrable systems are the exception. They have some underlying structure which helps us. Often, this structure consists of a number of symmetries:

An equation is integrable if has sufficiently many symmetries.

Each symmetry, in it infinitesimal form, defines a differential equation. Hence:

An equation is integrable if it is part of a sufficiently large family of compatible equations.

A common interpretation of "compatible" is given in terms of Hamiltonian mechanics.

Hamiltonian mechanics

Hamilton function

$$H : T^*Q \cong \mathbb{R}^{2N}(q,p) \to \mathbb{R}$$

Dynamics given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Flow consists of symplectic maps and preserves H.

Example If we take

$$H(q,p) = \frac{1}{2m}p^2 + V(q),$$

this yields

$$\dot{q}_i = \frac{1}{m}p, \qquad \dot{p}_i = -\frac{\partial V}{\partial q_i},$$

hence $m\ddot{q} = -\nabla V(q)$.

Poisson bracket of two functions on T^*Q:

$$\{f,g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right)$$

Dynamics of a Hamiltonian system:

$$\dot{q}_i = \{H, q_i\}$$
$$\dot{p}_i = \{H, p_i\}$$
$$\frac{\mathrm{d}}{\mathrm{d}t} f(q,p) = \{H, f\}$$

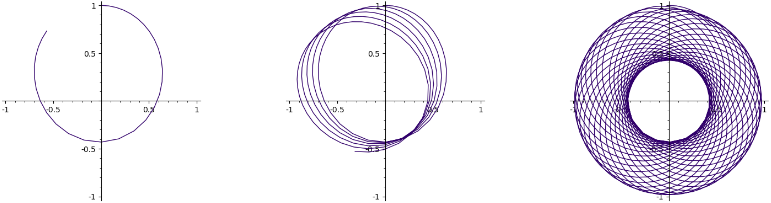
In particular:
f is conserved if and only if $\{H, f\} = 0$.

Liouville integrability

A Hamiltonian system with Hamilton function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \ldots H_N$ such that $\{H_i, H_j\} = 0$.

- Each H_i defines its own flow: N dynamical systems.
- Each H_i is a conserved quantity for all flows.
- Joint dynamics stay on $\{H_i = \text{const}\}$. If compact, this is a torus.

  Example: central force in the plane:



- Dynamics on these tori are linear in action-angle variables.
- The flows commute: $\phi^t_{H_i} \circ \phi^s_{H_j} = \phi^s_{H_j} \circ \phi^t_{H_i}$.

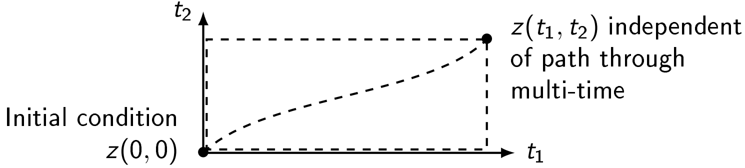
Two commuting flows

Let $z = (q, p)$. Consider two Hamiltonian ODEs

$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_1} = \{H_1(z), f(z)\}$$
$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_2} = \{H_2(z), f(z)\}$$

with $\{H_1, H_2\} = 0$

The flows commute, meaning that evolution can be parameterised by the (t_1, t_2) plane, called multi-time.



Additional commuting equations can be accommodated by increasing the dimension of multi-time: $\mathbb{R}^n$ instead of $\mathbb{R}^2$.

Lagrangian mechanics

Lagrange function $L : TQ \cong \mathbb{R}^{2N} \to \mathbb{R} : (q, q_t) \mapsto L(q, q_t)$

Dynamics follows curves which are minimizers (critical points) of the action

$$\int_a^b L(q, q_t)\,\mathrm{d}t \qquad \text{with fixed boundary values } q(a) \text{ and } q(b).$$

Minimizers satisfy the Euler-Lagrange (EL) equation $\quad \dfrac{\partial L}{\partial q} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial q_t} = 0$

Proof. Consider an arbitrary variation δq:

$$\delta \int_a^b L\,\mathrm{d}t = \int_a^b \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial q_t}\delta q_t \right) \mathrm{d}t$$

Integration by parts yields

$$\delta \int_a^b L\,\mathrm{d}t = \int_a^b \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q_t} \right) \delta q\,\mathrm{d}t + \left[\frac{\partial L}{\partial q_t}\delta q \right]_a^b$$

EL follows because $\delta q(a) = \delta q(b) = 0$ and δq is arbitrary inside (a, b). ∎

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Lagrangian formulation of Liouville integrable system

On the Hamiltonian side, commutativity is implied by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Suppose we have Lagrange functions L_i associated to H_i.

Variational (“Pluri-Lagrangian”/”Lagrangian multiform”) principle

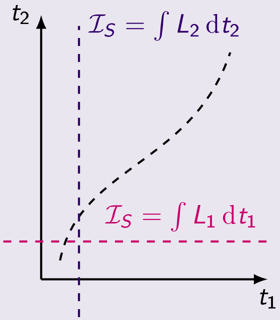
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q] \, \mathrm{d}t_i.$$

Look for dynamical variables $q(t_1, \ldots, t_N)$ such that the action

$$\mathcal{I}_S = \int_S \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve S in multi-time $\mathbb{R}^N$



Multi-time Euler-Lagrange equations

Assume that

$$L_1[q] = L_1(q, q_{t_1}),$$
$$L_i[q] = L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1$$

The multi-time Euler-Lagrange equations for $\mathcal{L} = \sum_i L_i[q]\,\mathrm{d}t_i$ are

Usual Euler-Lagrange equations: $$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_{t_i}} = 0$$

Usual EL wrt to alien derivatives: $$\frac{\partial L_i}{\partial q_{t_1}} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_{t_1 t_i}} = 0, \quad i \neq 1$$

Additional conditions: $$\frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}}$$

Example: Kepler Problem

The classical Lagrangian

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e \qquad (e \text{ fixed unit vector})$$

into a Lagrangian 1-form $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$.

Multi-time Euler-Lagrange equations:

$$\begin{aligned}
\frac{\partial L_1}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L_1}{\partial q_{t_1}} = 0 \quad &\Rightarrow \quad q_{t_1 t_1} = -\frac{q}{|q|^3} && \text{(Keplerian motion)} \\
\frac{\partial L_2}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_2}\frac{\partial L_2}{\partial q_{t_2}} = 0 \quad &\Rightarrow \quad q_{t_1 t_2} = e \times q_{t_1} && \\
\frac{\partial L_2}{\partial q_{t_1}} - \frac{\mathrm{d}}{\mathrm{d}t_2}\frac{\partial L_2}{\partial q_{t_1 t_2}} = 0 \quad &\Rightarrow \quad q_{t_2} = e \times q && \text{(Rotation)} \\
\frac{\partial L_1}{\partial q_{t_1}} = \frac{\partial L_2}{\partial q_{t_2}} \quad &\Rightarrow \quad q_{t_1} = q_{t_1} &&
\end{aligned}$$

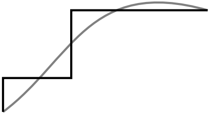
Derivation of the multi-time Euler-Lagrange equations

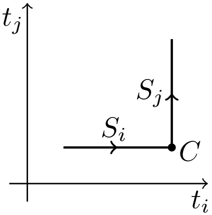
Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[q] \, \mathrm{d}t_i$, with

$$\begin{aligned} L_1[q] &= L_1(q, q_{t_1}), \\ L_i[q] &= L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1 \end{aligned}$$

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.



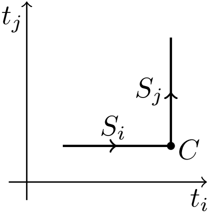


Variations are local, so it is sufficient to look at an L-shaped curve $S = S_i \cup S_j$.

Derivation of the multi-time Euler-Lagrange equations

On one of the straight pieces, S_i $(i \neq 1)$, we get

$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \left(\frac{\partial L_i}{\partial q}\delta q + \frac{\partial L_i}{\partial q_{t_1}}\delta q_{t_1} + \frac{\partial L_i}{\partial q_{t_i}}\delta q_{t_i} \right) \mathrm{d}t_i$$



Integration by parts (wrt t_i only) yields

$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \left(\left(\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_{t_i}} \right) \delta q + \frac{\partial L_i}{\partial q_{t_1}}\delta q_{t_1} \right) \mathrm{d}t_i + \frac{\partial L_i}{\partial q_{t_i}}\delta q \bigg|_C$$

Since p is an interior point of the curve, we cannot set $\delta q(C) = 0$!

Arbitrary δq and δq_{t_1}, so we find:

Multi-time Euler-Lagrange equations

$$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_{t_i}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_1}} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_{t_1 t_i}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}}$$

Higher order Lagranigans $L_i[q] = L_i(q, q_{t_i}, q_{t_i t_j}, \ldots)$

For a string $I = t_{i_1} \ldots t_{i_k}$ of time variables, denote the corresponding derivative by q_I.

If I is empty then $q_I = q$.

Denote by $\dfrac{\delta_i}{\delta q_I}$ the variational derivative in the direction of t_i wrt q_I:

$$\begin{aligned}\frac{\delta_i L_i}{\delta q_I} &= \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial L_i}{\partial q_{It_i^\alpha}} \\ &= \frac{\partial L_i}{\partial q_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{It_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_i}{\partial q_{It_i^2}} - \ldots\end{aligned}$$

Multi-time Euler-Lagrange equations

Usual Euler-Lagrange equations: $\dfrac{\delta_i L_i}{\delta q_I} = 0 \qquad \forall I \not\ni t_i,$

Additional conditions: $\dfrac{\delta_i L_i}{\delta q_{It_i}} = \dfrac{\delta_j L_j}{\delta q_{It_j}} \qquad \forall I,$

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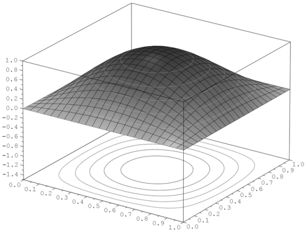
Variational principle for PDEs ($d = 2$)

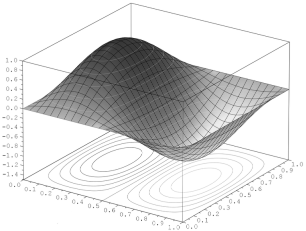
Pluri-Lagrangian principle

Given a 2-form

$$\mathcal{L}[q] = \sum_{i,j} L_{ij}[q]\, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $q : \mathbb{R}^N \to \mathbb{R}$, such that $\displaystyle\int_S \mathcal{L}[q]$ is critical on all smooth surfaces S in multi-time $\mathbb{R}^N$, w.r.t. variations of q.





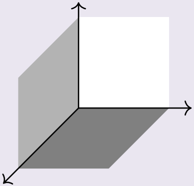
Multi-time EL equations

for $\mathcal{L}[q] = \sum_{i,j} L_{ij}[q]\, \mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{I t_j}} = \frac{\delta_{ik} L_{ik}}{\delta q_{I t_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{I t_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta q_{I t_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta q_{I t_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial q_{I t_i^\alpha t_j^\beta}}$$

Example: Potential KdV hierarchy

(Notation: u instead of q for the dependent variable)

$u_{t_2} = Q_2 = u_{xxx} + 3u_x^2,$

$u_{t_3} = Q_3 = u_{xxxxx} + 10u_x u_{xxx} + 5u_{xx}^2 + 10u_x^3,$

where we identify $t_1 = x$.

The differentiated equations $u_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x} Q_i$ are Lagrangian with

$$L_{12} = \frac{1}{2} u_x u_{t_2} - \frac{1}{2} u_x u_{xxx} - u_x^3,$$

$$L_{13} = \frac{1}{2} u_x u_{t_3} - \frac{1}{2} u_{xxx}^2 + 5 u_x u_{xx}^2 - \frac{5}{2} u_x^4.$$

A suitable coefficient L_{23} of

$$\mathcal{L} = L_{12}\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}\,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

can be found (nontrivial task!) in the form

$$L_{23} = \frac{1}{2}(u_{t_3} Q_2 - u_{t_2} Q_3) + p_{23}.$$

Example: Potential KdV hierarchy

- The equations $\dfrac{\delta_{12} L_{12}}{\delta u} = 0$ and $\dfrac{\delta_{13} L_{13}}{\delta u} = 0$ yield
$$u_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x} Q_2 \qquad \text{and} \qquad u_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x} Q_3.$$
- The equations $\dfrac{\delta_{12} L_{12}}{\delta u_x} = \dfrac{\delta_{32} L_{32}}{\delta u_{t_3}}$ and $\dfrac{\delta_{13} L_{13}}{\delta u_x} = \dfrac{\delta_{23} L_{23}}{\delta u_{t_2}}$ yield
$$u_{t_2} = Q_2 \qquad \text{and} \qquad u_{t_3} = Q_3,$$
the evolutionary equations!
- All other multi-time EL equations are consequences of these.

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Exterior derivative of $\mathcal{L}$

Revisit the Kepler problem: $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$ with

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e \qquad (e \text{ fixed unit vector})$$

Multi-time Euler-Lagrange equations:

$$q_{t_1 t_1} = -\frac{q}{|q|^3}$$

$$q_{t_2} = e \times q$$

Coefficient of $\mathrm{d}\mathcal{L}$

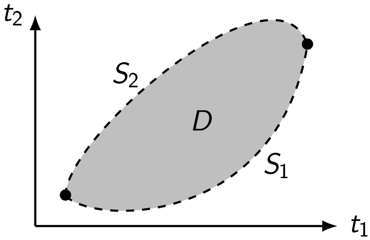
$$\frac{\mathrm{d}L_2}{\mathrm{d}t_1} - \frac{\mathrm{d}L_1}{\mathrm{d}t_2} = \left(q_{t_1 t_1} + \frac{q}{|q|^3} \right) (q_{t_2} - e \times q)$$

Observation (also for PDEs): $\mathrm{d}\mathcal{L}$ often has a "double zero" on solutions.

Interpretation of closedness condition

If $\mathrm{d}\mathcal{L} = 0$, then the action is invariant wrt variations in geometry

Deforming the curve (surface) of integration leaves action invariant.



By Stokes' theorem:

$$\int_{S_1} \mathcal{L} - \int_{S_2} \mathcal{L} = \int_D \mathrm{d}\mathcal{L} = 0$$

Hamiltonian formulation and $\mathrm{d}\mathcal{L}$

We can pass between the pluri-Lagrangian and Hamiltonian formalisms for 1-form and 2-forms.

Lemma ($\mathrm{d}\mathcal{L}$ for 1-forms)

On solutions there holds $\dfrac{\mathrm{d}L_j}{\mathrm{d}t_i} - \dfrac{\mathrm{d}L_i}{\mathrm{d}t_j} = \{H_j, H_i\}$.

It follows that:

Theorem

The Hamiltonians are in involution if and only if $\mathrm{d}\mathcal{L} = 0$ on solutions.

A similar result holds for 2-forms (and presumably for higher forms)

Variational Symmetries and $\mathrm{d}\mathcal{L}$

$\mathrm{d}\mathcal{L} = 0$ expresses that flows are variational symmetries of each other

$$\mathrm{d}\left(\sum_i L_i \,\mathrm{d}t_i\right) = 0 \Rightarrow \frac{\mathrm{d}L_k}{\mathrm{d}t_j} = \frac{\mathrm{d}L_j}{\mathrm{d}t_k}$$

$\Rightarrow$ t_j-flow changes L_k by a t_k-derivative

$$\Rightarrow \partial_j \int_a^b L_k \,\mathrm{d}t_k = \int_a^b \frac{\mathrm{d}L_j}{\mathrm{d}t_k}\,\mathrm{d}t_k = [L_j]_a^b = \mathrm{const}$$

Adding a constant to the action does not change the dynamics, hence ∂_j is a variational symmetry.

A similar result holds for higher forms.

We can use variational symmetries to construct Lagrangian multiforms.

Non-abelian symmetry groups

Not all symmetries commute with each other.

In the Kepler problem, the vector fields generating rotations satisfy

$$[\partial_1, \partial_2] = -\partial_3, \qquad [\partial_2, \partial_3] = -\partial_1 \qquad [\partial_3, \partial_1] = -\partial_2.$$

Even if a system is integrable (and especially if it is "super-integreable") the commuting Hamiltonian vector fields do not capture the symmetries in full.

Multiforms on Lie groups

If a system has symmetry group G, we can use the Lie group $\mathbb{R} \times G$ as multi-time.

Now $\mathrm{d}\mathcal{L} = 0$ relates the Poisson brackets to the Lie algebra of G.

In the special case where $G = \mathbb{R}^N$, this implies our earlier observation that $\mathrm{d}\mathcal{L} = 0 \iff \{H_i, H_j\} = 0$.

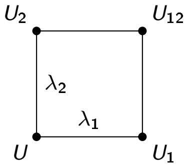
Multiforms are not just a tool in integrability, but a unified desciption of a system and its symmetries in general.

Discrete and semi-discrete multiforms

The same variational principle can be formulated in the discrete setting. (That is where it started!)

- It applies to fully discrete equations like the lattice potential KdV equation:

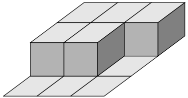
$$(U - U_{12})(U_1 - U_2) - \lambda_1 + \lambda_2 = 0.$$

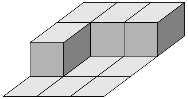


For some discrete 2-form

$$\mathcal{L}(\square_{ij}) = \mathcal{L}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j),$$

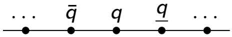
the action $\displaystyle\sum_{\square \in \Gamma} \mathcal{L}(\square)$ should be critical on all 2-surfaces Γ in $\mathbb{Z}^N$ simultaneously.





- It applies to semi-discrete equations like the Toda lattice:

$$q_{t_1 t_1} = \exp(\bar{q} - q) - \exp(q - \underline{q}).$$



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Summary

- The Lagrangian multiform (or pluri-Lagrangian) principle describes symmetries and integrability.

  Applies to ODEs and PDEs, discrete and continuous.

- Closedness of the Lagrangian form, i.e. $\mathrm{d}\mathcal{L} = 0$, is related to variational symmetries (Noether) and Poisson brackets.

- Some open questions:
  - Multiforms as a tool for construction solutions.
  - Full development for semi-discrete systems
  - Better understanding of application to gauge theory
  - Application to quantum integrable systems, path integrals, ...

Selected references

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Thank you for your attention!