A Lagrangian perspective on integrability

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1. Introduction

Many integrable systems can be understood as hierarchies, consisting of the integrable equation and its symmetries, with a (bi-)Hamiltonian structure. This poster presents a lesser-known Lagrangian perspective on integrability.

Known by the names Lagrangian multiforms and pluri-Lagrangian systems, it captures a whole hierarchy in a single variational principle. It can be used to describe many types of integrable systems (fully discrete, differential-difference, and continuous equations, of various dimensions). In some cases it can be used to show surprising implications or equivalences between sets of integrable equations.

Boxes 2–5 present Lagrangian multiform (1-form) theory for integrable ODEs, starting from the perspective of Liouville integrability. Boxes 6–9 provide glimpses of multiform theory for PDEs and (semi-)discrete equations.

2. Liouville integrability

A Hamiltonian system with n degrees of freedom is Liouville integrable if its Hamilton function is part of a family of n independent functions $H_1, \dots, H_n$ that satisfy

$$\{H_i, H_j\} = 0.$$

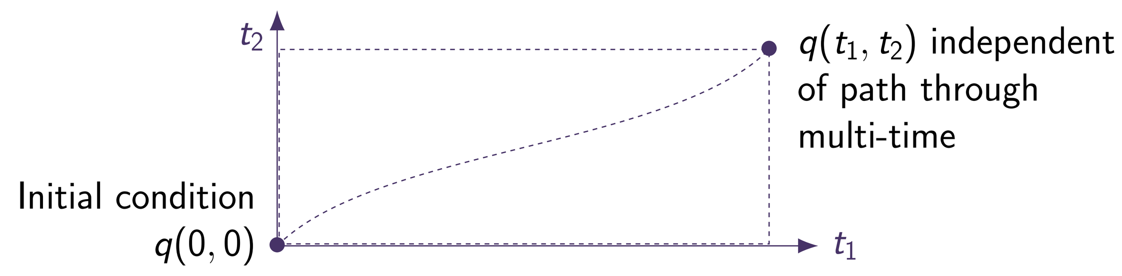
Properties.

- $H_1, \dots, H_n$ are conserved quantities.
- Each H_i can be used as a Hamilton function and defines a dynamical system.
- The flows of these systems commute with each other, so we can consider simultaneous solutions $q(t_1, \dots, t_n)$ of $\frac{\partial q}{\partial t_i} = \{H_i, q\}$.
- Dynamics stays on a common level set of the $H_1, \dots, H_n$. If this level set is compact, it is a topological torus.
- Dynamics are linear in action-angle coordinates.

Can we characterise Liouville integrability from the Lagrangian side, i.e. using a variational principle?

3. Variational principle in multi-time

Minimal example: consider two commuting ODEs. Associate times t_1 and t_2 to the respective flows. The evolution is parameterised by the (t_1, t_2)-plane, called multi-time.



(Take $\mathbb{R}^n$ instead of $\mathbb{R}^2$ to include additional commuting flows.)

Usual Lagrangian description. Infinitesimal variations of q leave

$$\int L_1(q, q_{t_1})\,\mathrm{d}t_1 \quad \text{and} \quad \int L_2(q, q_{t_1}, q_{t_2})\,\mathrm{d}t_2$$

invariant. (Subscripts denote partial derivatives.)

Pluri-Lagrangian principle. Consider the 1-form $\mathcal{L}[q] = L_1(q, q_{t_1})\,\mathrm{d}t_1 + L_2(q, q_{t_1}, q_{t_2})\,\mathrm{d}t_2$. For every curve $\gamma : [0,1] \to \mathbb{R}^2$ in multi-time and every variation v with suitable boundary conditions:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_\gamma \mathcal{L}[q + \varepsilon v]\Big|_{\varepsilon=0} = 0.$$

4. Multi-time Euler-Lagrange equations

Taking variations of q leads to the usual Euler-Lagrange equations

$$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_{t_i}} = 0 \qquad \text{(EL1)}$$

as well as

$$\frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}} \quad \text{and} \quad \frac{\partial L_i}{\partial q_{t_j}} = 0 \text{ if } i \neq j. \qquad \text{(EL2)}$$

(For Lagrangians depending on higher derivatives of q, natural generalisations hold.)

Theorem. Equations (EL1)-(EL2) characterise fields q satisfying the pluri-Lagrangian principle. [8, 9]

Theorem. The following are equivalent:

- $\mathrm{d}\mathcal{L} = 0$ when evaluated on solutions q to the multi-time Euler-Lagrange equations (EL1)-(EL2).
- Lagrangian multiform principle: on solutions to (EL1)-(EL2), the action is critical with respect to variations of the curve γ (as opposed to variations of q only). [3, 4]
- Vanishing Poisson brackets between the corresponding Hamiltonian functions. [11]

5. One-form example

The Kepler Lagrangian

$$L_1 = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

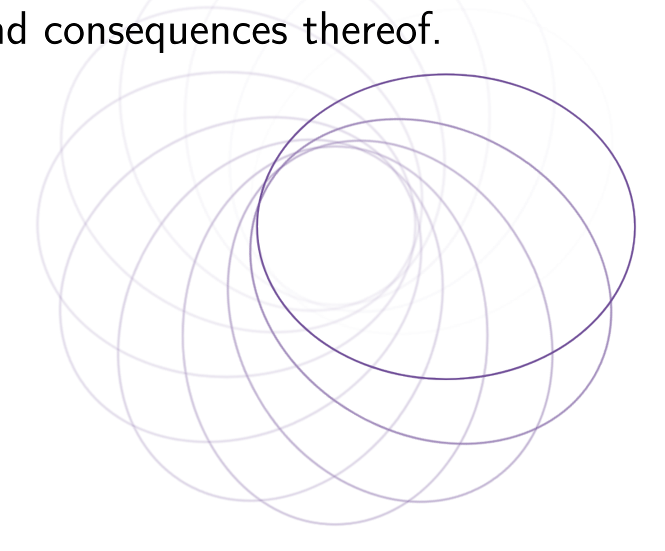
$$L_2 = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e$$

to form a 1-form $\mathcal{L} = L_1\,\mathrm{d}t_1 + L_2\,\mathrm{d}t_2$, yielding the multi-time EL equations

$$q_{t_1 t_1} = -\frac{q}{|q|^3}, \quad \text{(inverse square law)}$$

$$q_{t_2} = e \times q \quad \text{(rotation)}$$

and consequences thereof.



6. Partial differential equations

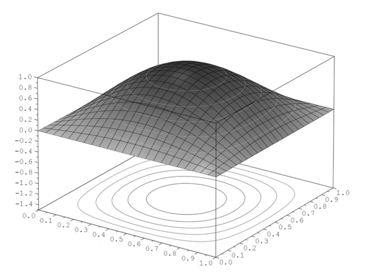
Hierarchies of PDEs share their space variables but have separate time variables in multi-time.

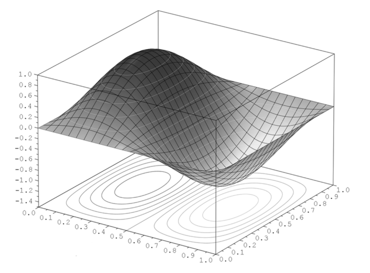
For 2-dimensional PDEs, we consider a 2-form

$$\mathcal{L} = \sum_{i,j} L_{ij}[q]\,\mathrm{d}t_i \wedge \mathrm{d}t_j.$$

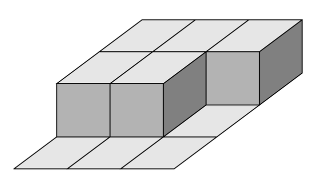
Solutions are fields q such that the integral of $\mathcal{L}$ is critical over every surface in multi-time.

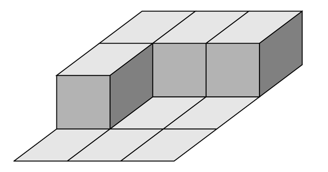
Lagrangian 2-forms are known for potential KdV [9], modified KdV [5], AKNS [7], ...





7. Difference equations





For difference equations on elementary squares of $\mathbb{Z}^2$, the action sum of a discrete 2-form should be critical on every discrete surface in $\mathbb{Z}^3$.

This perspective has played an important role in the study of integrable partial difference equations [1, 3, 4]. Continuum limits were studied in [10].

8. Two-form Example

Consider the first two equations of the potential KdV hierarchy:

$$u_{t_2} = Q_2 = u_{xxx} + 3u_x^2,$$
$$u_{t_3} = Q_3 = u_{xxxxx} + 10u_x u_{xxx} + 5u_{xx}^2 + 10u_x^3,$$

where we set $t_1 = x$. These are Lagrangian, in a weak sense, with

$$L_{12} = \frac{1}{2}u_x u_{t_2} - \frac{1}{2}u_x u_{xxx} - u_x^3,$$
$$L_{13} = \frac{1}{2}u_x u_{t_3} - \frac{1}{2}u_{xxx}^2 + 5u_x u_{xx}^2 - \frac{5}{2}u_x^4.$$

A suitable coefficient L_{23} of

$$\mathcal{L} = L_{12}\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}\,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

can be found (nontrivial task!) in the form

$$L_{23} = \frac{1}{2}(u_{t_3}Q_2 - u_{t_2}Q_3) + p_{23}.$$

The usual Euler-Lagrange equations of L_{12} and L_{13} yield

$$u_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x}Q_2 \quad \text{and} \quad u_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x}Q_3,$$

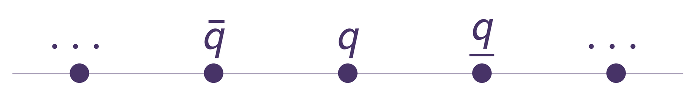
but the multi-time Euler-Lagrange equations of $\mathcal{L}$ consist of

$$u_{t_2} = Q_2 \quad \text{and} \quad u_{t_3} = Q_3,$$

and consequences thereof.

9. Semi-discrete equations

Consider a sequence of particles on a line:



(One discrete dimension, many continuous times.)

Toda lattice: exponential forces between neighbours,

$$q_{t_1 t_1} = \exp(\bar{q} - q) - \exp(q - \underline{q}). \qquad \text{(T1)}$$

(T1) is part of a hierarchy. Its next member is

$$q_{t_2} = q_{t_1}^2 + \exp(\bar{q} - q) + \exp(q - \underline{q}). \qquad \text{(T2)}$$

It possesses a semi-discrete Lagrangian 2-form with the following coefficients ("0" stands for the discrete direction):

$$L_{01} = \tfrac{1}{2}q_{t_1}^2 - \exp(\bar{q} - q),$$
$$L_{02} = q_{t_1}q_{t_2} - \tfrac{1}{3}q_{t_1}^3 - (q_{t_1} + \bar{q}_{t_1})\exp(\bar{q} - q),$$
$$L_{12} = \tfrac{1}{4}\left(\bar{q}_{t_2} - \bar{q}_{t_1 t_1} - \bar{q}_{t_1}^2\right)^2.$$

The multi-time Euler-Lagrange equations are (T1)–(T2) and

$$\tfrac{1}{2}q_{t_2 t_2} - q_{t_1 t_1}q_{t_2} - 2q_{t_1 t_2}q_{t_1} - \tfrac{1}{2}q_{t_1 t_1 t_1 t_1} + 3q_{t_1}^2 q_{t_1 t_1} = 0.$$

The multiform produces a scalar PDE at a single lattice site. This is in contrast to known PDEs associated to the Toda hierarchy involving two variables (two lattice sites). [6]

10. Outlook

The theory of pluri-Lagrangian systems and Lagrangian multiforms shows that a variational description of integrable systems is possible.

In certain contexts, such as discretisation, the Lagrangian point of view has clear advantages.

Some topics of ongoing research are:

- Lagrangian multiforms for symmetries that do not commute with each other, by replacing euclidean multi-time with a Lie group(oid). In particular, this is relevant to super-integrable systems. [2]
- Extending the theory to systems with infinite-dimensional symmetry groups (setting of Noether's second theorem), in particular to gauge theories of physics.
- Generalising the Toda example: relating integrable differential-difference equations and PDEs.
- Using connections between $\mathrm{d}\mathcal{L}$ and the multi-time Euler-Lagrange equations to relate PDEs of different dimensionality.

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