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A variational principle for integrable systems, symmetries, and discretisation

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Integrable systems

Most nonlinear differential equations are impossible to solve explicitly.

Integrable systems are the exception. They have some underlying structure which helps us.

Often, this structure consists of a number of symmetries:

An equation is integrable if has sufficiently many symmetries.

Each symmetry, in it infinitesimal form, defines a differential equation. Hence:

An equation is integrable if it is part of a sufficiently large family of compatible equations.

A common interpretation of “compatible” is given in terms of Hamiltonian mechanics.

Hamiltonian mechanics

Hamilton function

$$H : \mathbb{R}^{2N} \cong T^*Q \to \mathbb{R} : \\ (q,p) \mapsto H(q,p)$$

Typically

$$H(q,p) = \frac{1}{2m}p^2 + U(q)$$

Dynamics given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Flow consists of symplectic maps and preserves H.

Poisson bracket of two functions on T^*Q:

$$\{f,g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right)$$

Dynamics of a Hamiltonian system:

$$\dot{q}_i = \{H, q_i\}, \\ \dot{p}_i = \{H, p_i\}, \\ \frac{\mathrm{d}}{\mathrm{d}t} f(q,p) = \{H, f\}.$$

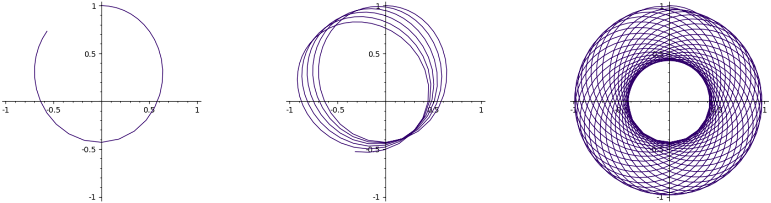
In particular: f is conserved if and only if $\{H,f\} = 0$.

Liouville integrability

A Hamiltonian system with Hamilton function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \ldots H_N$ such that $\{H_i, H_j\} = 0$.

- Each H_i defines its own flow: N dynamical systems.
- Each H_i is a conserved quantity for all flows.
- Joint dynamics stay on $\{H_i = \text{const}\}$. If compact, this is a torus.

  Example: central force in the plane:



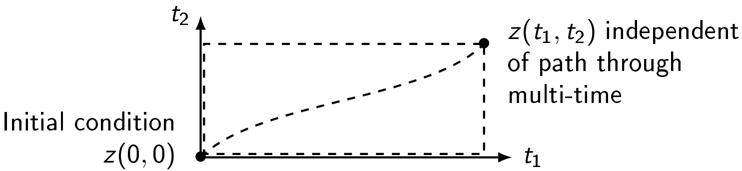
- Dynamics on these tori are linear in action-angle variables.
- The flows commute: $\phi^t_{H_i} \circ \phi^s_{H_j} = \phi^s_{H_j} \circ \phi^t_{H_i}$.

Two commuting flows

Let $z = (q,p)$. Consider two Hamiltonian ODEs

$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_1} = \{H_1(z), f(z)\} \qquad \text{with } \{H_1, H_2\} = 0$$
$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_2} = \{H_2(z), f(z)\}$$

The flows commute, meaning that evolution can be parameterised by the (t_1, t_2) plane, called multi-time.



Additional commuting equations can be accommodated by increasing the dimension of multi-time: $\mathbb{R}^n$ instead of $\mathbb{R}^2$.

Lagrangian mechanics

Lagrange function $L : TQ \cong \mathbb{R}^{2N} \to \mathbb{R} : (q, q_t) \mapsto L(q, q_t)$

Dynamics follows curves which are minimizers (critical points) of the action

$$\int_a^b L(q, q_t)\,\mathrm{d}t \qquad \text{with fixed boundary values } q(a) \text{ and } q(b).$$

Minimizers satisfy the Euler-Lagrange (EL) equation $\quad \dfrac{\partial L}{\partial q} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial q_t} = 0$

Proof. Consider an arbitrary variation δq:

$$\delta \int_a^b L\,\mathrm{d}t = \int_a^b \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial q_t}\delta q_t \right) \mathrm{d}t$$

Integration by parts yields

$$\delta \int_a^b L\,\mathrm{d}t = \int_a^b \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q_t} \right) \delta q\,\mathrm{d}t + \left[\frac{\partial L}{\partial q_t}\delta q \right]_a^b$$

EL follows because $\delta q(a) = \delta q(b) = 0$ and δq is arbitrary inside (a, b). ∎

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Lagrangian formulation of commuting flows

On the Hamiltonian side, commutativity is implied by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Suppose we have Lagrange functions L_i associated to H_i.

Variational ("Pluri-Lagrangian"/"Lagrangian multiform") principle

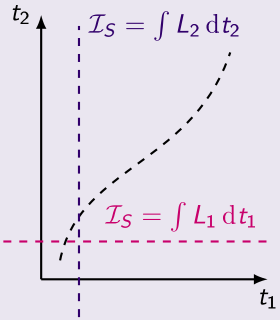
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q] \, \mathrm{d}t_i .$$

Look for dynamical variables $q(t_1, \ldots, t_N)$ such that the action

$$\mathcal{I}_S = \int_S \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve S in multi-time $\mathbb{R}^N$



Multi-time Euler-Lagrange equations

Assume that

$$L_1[q] = L_1(q, q_{t_1}),$$
$$L_i[q] = L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1$$

The multi-time Euler-Lagrange equations for $\mathcal{L} = \sum_i L_i[q]\,\mathrm{d}t_i$ are

Usual Euler-Lagrange equations: $$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_{t_i}} = 0$$

Usual EL wrt to alien derivatives: $$\frac{\partial L_i}{\partial q_{t_1}} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_{t_1 t_i}} = 0, \quad i \neq 1$$

Additional conditions: $$\frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}}$$

Example: Kepler Problem

The classical Lagrangian

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e \qquad (e \text{ fixed unit vector})$$

into a Lagrangian 1-form $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$.

Multi-time Euler-Lagrange equations:

$$\frac{\partial L_1}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L_1}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_1 t_1} = -\frac{q}{|q|^3} \qquad \text{(Keplerian motion)}$$

$$\frac{\partial L_2}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_2}\frac{\partial L_2}{\partial q_{t_2}} = 0 \quad \Rightarrow \quad q_{t_1 t_2} = e \times q_{t_1}$$

$$\frac{\partial L_2}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_2} = e \times q \qquad \text{(Rotation)}$$

$$\frac{\partial L_1}{\partial q_{t_1}} = \frac{\partial L_2}{\partial q_{t_2}} \quad \Rightarrow \quad q_{t_1} = q_{t_1}$$

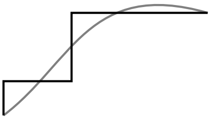
Derivation of the multi-time Euler-Lagrange equations

Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[q] \, \mathrm{d}t_i$, with

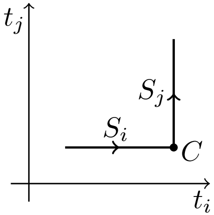
$$L_1[q] = L_1(q, q_{t_1}),$$
$$L_i[q] = L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1$$

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.



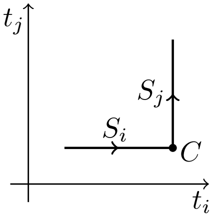
Variations are local, so it is sufficient to look at an L-shaped curve $S = S_i \cup S_j$.



Derivation of the multi-time Euler-Lagrange equations

On one of the straight pieces, S_i $(i \neq 1)$, we get

$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \left(\frac{\partial L_i}{\partial q} \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} + \frac{\partial L_i}{\partial q_{t_i}} \delta q_{t_i} \right) \mathrm{d}t_i$$



Integration by parts (wrt t_i only) yields

$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \left(\left(\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} \right) \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} \right) \mathrm{d}t_i + \frac{\partial L_i}{\partial q_{t_i}} \delta q \bigg|_C$$

Since p is an interior point of the curve, we cannot set $\delta q(C) = 0$!

Arbitrary δq and δq_{t_1}, so we find:

Multi-time Euler-Lagrange equations

$$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_1}} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_1 t_i}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}}$$

Higher order Lagranigans $L_i[q] = L_i(q, q_{t_i}, q_{t_i t_j}, \ldots)$

For a string $I = t_{i_1} \ldots t_{i_k}$ of time variables, denote the corresponding derivative by q_I.

If I is empty then $q_I = q$.

Denote by $\dfrac{\delta_i}{\delta q_I}$ the variational derivative in the direction of t_i wrt q_I:

$$\begin{aligned}\frac{\delta_i L_i}{\delta q_I} &= \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial L_i}{\partial q_{I t_i^\alpha}} \\ &= \frac{\partial L_i}{\partial q_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{I t_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_i}{\partial q_{I t_i^2}} - \ldots\end{aligned}$$

Multi-time Euler-Lagrange equations

Usual Euler-Lagrange equations: $\dfrac{\delta_i L_i}{\delta q_I} = 0 \qquad \forall I \not\ni t_i,$

Additional conditions: $\dfrac{\delta_i L_i}{\delta q_{I t_i}} = \dfrac{\delta_j L_j}{\delta q_{I t_j}} \qquad \forall I,$

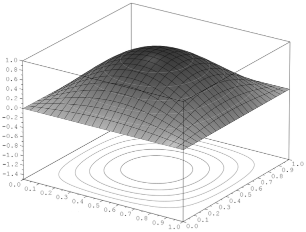
Variational principle for PDEs ($d = 2$)

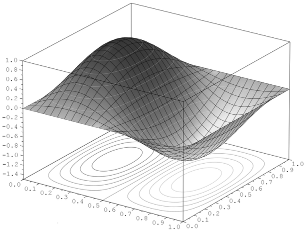
Pluri-Lagrangian principle

Given a 2-form

$$\mathcal{L}[q] = \textstyle\sum_{i,j} L_{ij}[q] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $q : \mathbb{R}^N \to \mathbb{R}$, such that $\displaystyle\int_S \mathcal{L}[q]$ is critical on all smooth surfaces S in multi-time $\mathbb{R}^N$, w.r.t. variations of q.





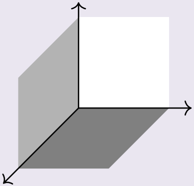
Multi-time EL equations

for $\mathcal{L}[q] = \sum_{i,j} L_{ij}[q]\,\mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta q_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta q_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta q_{It_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial q_{It_i^\alpha t_j^\beta}}$$

Example: Potential KdV hierarchy

(Notation: u instead of q for the dependent variable)

$u_{t_2} = Q_2 = u_{xxx} + 3u_x^2$,

$u_{t_3} = Q_3 = u_{xxxxx} + 10u_x u_{xxx} + 5u_{xx}^2 + 10u_x^3$,

where we identify $t_1 = x$.

The differentiated equations $u_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x} Q_i$ are Lagrangian with

$$\begin{aligned} L_{12} &= \frac{1}{2} u_x u_{t_2} - \frac{1}{2} u_x u_{xxx} - u_x^3, \\ L_{13} &= \frac{1}{2} u_x u_{t_3} - \frac{1}{2} u_{xxx}^2 + 5u_x u_{xx}^2 - \frac{5}{2} u_x^4. \end{aligned}$$

A suitable coefficient L_{23} of

$$\mathcal{L} = L_{12}\, \mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}\, \mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}\, \mathrm{d}t_2 \wedge \mathrm{d}t_3$$

can be found (nontrivial task!) in the form

$$L_{23} = \frac{1}{2}(u_{t_3} Q_2 - u_{t_2} Q_3) + p_{23}.$$

Example: Potential KdV hierarchy

- The equations $\dfrac{\delta_{12} L_{12}}{\delta u} = 0$ and $\dfrac{\delta_{13} L_{13}}{\delta u} = 0$ yield

$$u_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x} Q_2 \qquad \text{and} \qquad u_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x} Q_3 .$$

- The equations $\dfrac{\delta_{12} L_{12}}{\delta u_x} = \dfrac{\delta_{32} L_{32}}{\delta u_{t_3}}$ and $\dfrac{\delta_{13} L_{13}}{\delta u_x} = \dfrac{\delta_{23} L_{23}}{\delta u_{t_2}}$ yield

$$u_{t_2} = Q_2 \qquad \text{and} \qquad u_{t_3} = Q_3 ,$$

  the evolutionary equations!

- All other multi-time EL equations are consequences of these.

Exterior derivative of $\mathcal{L}$

Revisit the Kepler problem: $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$ with

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e \qquad (e \text{ fixed unit vector})$$

Multi-time Euler-Lagrange equations:

$$q_{t_1 t_1} = -\frac{q}{|q|^3}$$

$$q_{t_2} = e \times q$$

Coefficient of $\mathrm{d}\mathcal{L}$

$$\frac{\mathrm{d}L_2}{\mathrm{d}t_1} - \frac{\mathrm{d}L_1}{\mathrm{d}t_2} = \left(q_{t_1 t_1} + \frac{q}{|q|^3} \right) (q_{t_2} - e \times q)$$

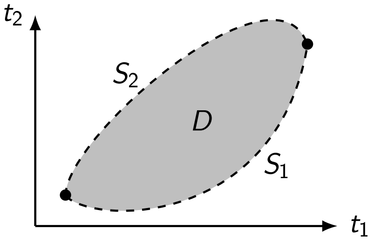
Observation (also for PDEs): $\mathrm{d}\mathcal{L}$ often has a “double zero” on solutions.

$\mathrm{d}\mathcal{L} = 0$ sets a Lagrangian multiform apart from a pluri-Lagrangian system.

Interpretation of closedness condition

If $\mathrm{d}\mathcal{L} = 0$, then the action is invariant wrt variations in geometry

Deforming the curve (surface) of integration leaves action invariant.



$$\int_{S_1} \mathcal{L} - \int_{S_2} \mathcal{L} = \int_D \mathrm{d}\mathcal{L} = 0$$

Recall: in the "pluri-Lagrangian" variational principle, we only took variations of the dependent variable q, not of the curve through multi-time.

Hamiltonian formulation and $\mathrm{d}\mathcal{L}$

We can pass between the pluri-Lagrangian and Hamiltonian formalisms for 1-form and 2-forms.

Lemma ($\mathrm{d}\mathcal{L}$ for 1-forms)

On solutions there holds $\dfrac{\mathrm{d}L_j}{\mathrm{d}t_i} - \dfrac{\mathrm{d}L_i}{\mathrm{d}t_j} = \{H_j, H_i\}$.

It follows that:

Theorem

The Hamiltonians are in involution if and only if $\mathrm{d}\mathcal{L} = 0$ on solutions.

A similar result holds for 2-forms (and presumably for higher forms)

Variational Symmetries and $\mathrm{d}\mathcal{L}$

$\mathrm{d}\mathcal{L} = 0$ expresses that flows are variational symmetries of each other

$$\mathrm{d}\left(\sum_i L_i \, \mathrm{d}t_i \right) = 0 \Rightarrow \frac{\mathrm{d}L_k}{\mathrm{d}t_j} = \frac{\mathrm{d}L_j}{\mathrm{d}t_k}$$

$$\Rightarrow t_j\text{-flow changes } L_k \text{ by a } t_k\text{-derivative}$$

$$\Rightarrow \partial_j \int_a^b L_k \, \mathrm{d}t_k = \int_a^b \frac{\mathrm{d}L_j}{\mathrm{d}t_k} \, \mathrm{d}t_k = [L_j]_a^b = \mathrm{const}$$

Adding a constant to the action does not change the dynamics, hence ∂_j is a variational symmetry.

A similar result holds for higher forms.

We can use variational symmetries to construct Lagrangian multiforms.

Non-abelian symmetry groups

Not all symmetries commute with each other.

In the Kepler problem, the vector fields generating rotations satisfy

$$[\partial_1, \partial_2] = -\partial_3, \qquad [\partial_2, \partial_3] = -\partial_1 \qquad [\partial_3, \partial_1] = -\partial_2.$$

If $[\partial_i, \partial_j] \neq 0$, then there do not exists time variables t_i such that $\partial_i = \frac{\partial}{\partial t_i}$.

Even if a system is integrable (and especially if it is "super-integreable") the commuting Hamiltonian vector fields do not capture the symmetries in full.

If a system has symmetry group G, we can use the Lie group $\mathbb{R} \times G$ as multi-time.

Multiforms on Lie groups

Let ξ, ν be elements of its Lie algebra $\mathfrak{g}$. Then $\iota_\nu \mathcal{L}$, $\iota_\xi \mathcal{L}$ are coefficients of the Lagrangian 1-form.

Theorem

The following are equivalent

1. $\mathrm{d}\mathcal{L} = 0$ on solutions
2. Cross-differentiation Lagrangians gives Lagrangian for commutator:
$$\iota_{[\xi,\nu]}\mathcal{L} = \partial_\xi \iota_\nu \mathcal{L} - \partial_\nu \iota_\xi \mathcal{L}$$
3. Poisson Bracket gives Hamiltonian for commutator:
$$H_{[\xi,\nu]} = \{H_\xi, H_\nu\} \quad \text{or, equivalently} \quad [X_{H_\xi}, X_{H_\nu}] = X_{\{H_\xi, H_\nu\}}$$

In case $G = \mathbb{R}^N$, this implies our earlier observation $\mathrm{d}\mathcal{L} = 0 \Leftrightarrow \{H_i, H_j\} = 0$.

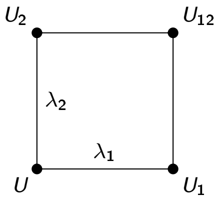
Multiforms are not just a tool in integrability, but a unified desciption of a system and its symmetries in general.

Contents

Quad equations

$$\mathcal{Q}(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2) = 0$$

- Subscripts of U denote lattice shifts.
- λ_1, λ_2 are parameters.
- Invariant under symmetries of the square, affine in each of U, U_1, U_2, U_{12}.



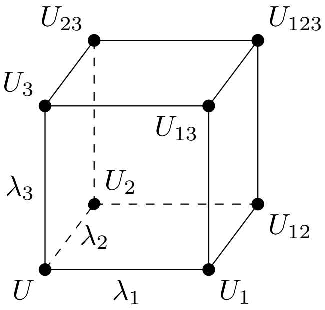
Discrete analogue of commuting flows:

Multi-dimensional consistency

The three ways of calculating U_{123}, using

$$\mathcal{Q}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) = 0,$$

and its shifts, give the same result.



Example: lattice potential KdV:

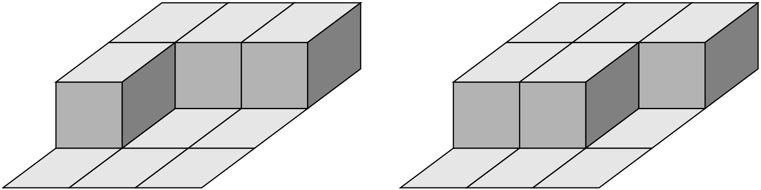
$$(U - U_{12})(U_1 - U_2) - \lambda_1 + \lambda_2 = 0$$

Variational principle for quad equations

For some discrete 2-form

$$\mathcal{L}(\square_{ij}) = \mathcal{L}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j),$$

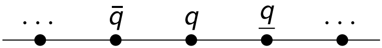
the action $\displaystyle\sum_{\square \in \Gamma} \mathcal{L}(\square)$ is critical on all 2-surfaces Γ in $\mathbb{Z}^N$ simultaneously.



The discrete and continuous variational principles are the same.

Semi-discrete systems

Consider particles on a line: 1 discrete dimension, many continuous times



Denote $\quad q_1 = q_{t_1} = \frac{\mathrm{d}q}{\mathrm{d}t_1}, \quad q_{11} = q_{t_1 t_1} = \frac{\mathrm{d}^2 q}{\mathrm{d}t_1^2}, \quad$ etc.

Toda lattice: exponential nearest-neighbour interaction

$$q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q}).$$

Part of a hierarchy. First symmetry:

$$q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q})$$

Toda lattice

Lagrangians ("0" for discrete direction)

$$L_{01} = \frac{1}{2} q_1^2 - \exp(\bar{q} - q)$$
$$L_{02} = q_1 q_2 - \frac{1}{3} q_1^3 - (q_1 + \bar{q}_1) \exp(\bar{q} - q)$$
$$L_{12} = -\frac{1}{4} \left(q_2 - q_{11} - q_1^2\right)^2$$

Euler-Lagrange equations:

$$\frac{\delta_{01} L_{01}}{\delta q} = 0 \qquad \rightarrow \qquad q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q})$$
$$\frac{\delta_{02} L_{02}}{\delta q_1} = 0 \qquad \rightarrow \qquad q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q})$$
$$\frac{\delta_{12} L_{12}}{\delta q} = 0 \qquad \rightarrow \qquad \frac{1}{2} q_{22} - q_{11} q_2 - 2 q_{12} q_1 - \frac{1}{2} q_{1111} + 3 q_1^2 q_{11} = 0$$

Lagrangian formalism produces a non-trivial PDE at a single lattice site.

Contents

Summary

- The Lagrangian multiform (or pluri-Lagrangian) principle describes symmetries and integrability.

  Applies to ODEs and PDEs, discrete and continuous.

- Closedness of the Lagrangian form, i.e. $\mathrm{d}\mathcal{L} = 0$, is related to variational symmetries (Noether) and Poisson brackets.

- Some open questions:
  - Multiforms as a tool for construction solutions.
  - Full development for semi-discrete systems

    Semi-discrete multiforms in geometric numerical integration?
    Geometric integrators are discrete maps with continuous symmetries.
  - Full understanding of general symmetry groups/groupoids
  - Applications to gauge theory?
  - Application to quantum integrable systems, path integrals, ...

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Thank you for your attention!