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# A variational principle for integrable systems

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# Contents

# Hamiltonian Systems

Hamilton function

$$H : T^*Q \cong \mathbb{R}^{2N} \to \mathbb{R} : (q,p) \mapsto H(q,p)$$

determines dynamics:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Canonical Poisson bracket of two functions of $T^*Q$:

$$\{f,g\} = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right)$$

Dynamics:

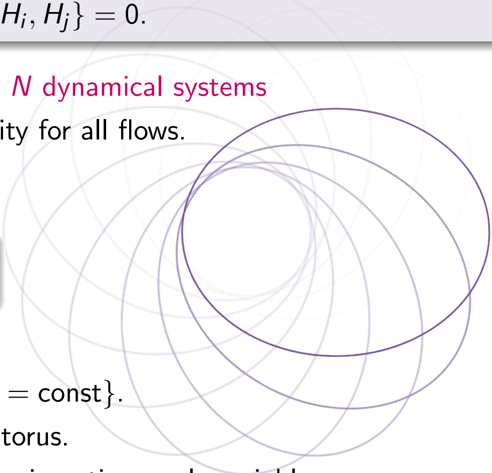
$$\frac{\mathrm{d}}{\mathrm{d}t} f(q,p) = \big\{ H(q,p), f(q,p) \big\}$$

Note that $f$ is a conserved quantity if and only if $\{H,f\} = 0$.

# Liouville integrability

A Hamiltonian system with Hamilton function $H : T^*Q \cong \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville integrable if there exist $N$ functionally independent functions $H = H_1, H_2, \ldots, H_N$ that satisfy $\{H_i, H_j\} = 0$.

- Each $H_i$ defines its own flow: $N$ dynamical systems
- Each $H_i$ is a conserved quantity for all flows.
- The flows commute.



Integrability $\approx$ being part of a large set of compatible equations

Consequences:

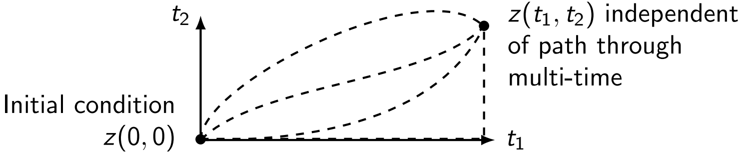
- State stays on a level set $\{H_i = \text{const}\}$.
- If compact, this level set is a torus.
- Dynamics on the torus is linear in action-angle variables.

# Two commuting flows

Let $z = (q, p)$. Consider two Hamiltonian ODEs

$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_1} = \{H_1(z), f(z)\} \qquad \text{with } \{H_1, H_2\} = 0$$
$$\frac{\mathrm{d}f(z)}{\mathrm{d}t_2} = \{H_2(z), f(z)\}$$

The flows commute, meaning that evolution can be parametrised by the $(t_1, t_2)$ plane, called multi-time.




Additional commuting equations can be accommodated by increasing the dimension of multi-time: $\mathbb{R}^n$ instead of $\mathbb{R}^2$.

# Lagrangian formulation of Liouville integrable system

On the Hamiltonian side, commutativity is implied by $\{H_i, H_j\} = 0$.

What about the Lagrangian side?

Suppose we have Lagrange functions $L_i$ associated to $H_i$.

## Pluri-Lagrangian (Lagrangian multi-form) principle for ODEs

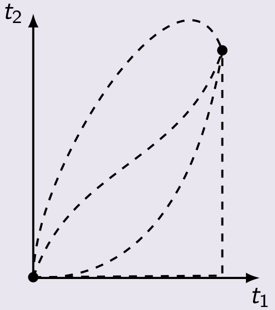
combine the $L_i$ into a Lagrangian 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q] \,\mathrm{d}t_i.$$

Look for dynamical variables $q(t_1, \ldots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[q]$$

is critical w.r.t. variations of $q$, simultaneously over every curve $\Gamma$ in multi-time $\mathbb{R}^N$



# Multi-time Euler-Lagrange equations

Assume that

$$L_1[q] = L_1(q, q_{t_1}) \qquad \text{and} \qquad L_i[q] = L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1$$

Then the multi-time Euler-Lagrange equations for $\mathcal{L} = \sum_i L_i[q] \, \mathrm{d}t_i$ are

Usual Euler-Lagrange equations: $$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} = 0$$

Usual EL wrt to alien derivatives: $$\frac{\partial L_i}{\partial q_{t_1}} - \cancel{\frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_1 t_i}}} = 0, \quad i \neq 1$$

Additional conditions: $$\frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}}$$

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Suris. Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geometric Mechanics, 2013

Suris, V. On the Lagrangian structure of integrable hierarchies. In: Advances in Discrete Differential Geometry, Springer. 2016.

# Example: Kepler Problem

The classical Lagrangian

$$L_1[q] = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e \qquad (e \text{ fixed unit vector})$$

into a Lagrangian 1-form $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$.

Multi-time Euler-Lagrange equations:

$$\frac{\partial L_1}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L_1}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_1 t_1} = -\frac{q}{|q|^3} \qquad \text{(Keplerian motion)}$$

$$\frac{\partial L_2}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_2}\frac{\partial L_2}{\partial q_{t_2}} = 0 \quad \Rightarrow \quad q_{t_1 t_2} = e \times q_{t_1}$$

$$\frac{\partial L_2}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_2} = e \times q \qquad \text{(Rotation)}$$

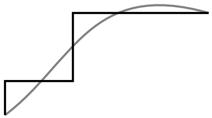
$$\frac{\partial L_1}{\partial q_{t_1}} = \frac{\partial L_2}{\partial q_{t_2}} \quad \Rightarrow \quad q_{t_1} = q_{t_1}$$

# Derivation of the multi-time Euler-Lagrange equations

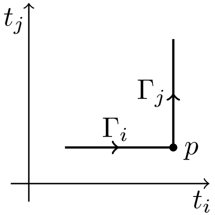
Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[q] \, \mathrm{d}t_i$

**Lemma**

If the action $\int_\Gamma \mathcal{L}$ is critical on all stepped curves $\Gamma$ in $\mathbb{R}^N$, then it is critical on all smooth curves.



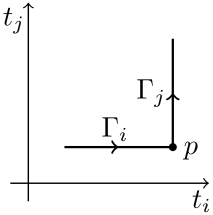
Variations are local, so it is sufficient to look at an L-shaped curve $\Gamma = \Gamma_i \cup \Gamma_j$.



# Derivation of the multi-time Euler-Lagrange equations

On one of the straight pieces, $\Gamma_i$ $(i \neq 1)$, we get

$$\delta \int_{\Gamma_i} L_i \,\mathrm{d}t_i = \int_{\Gamma_i} \left( \frac{\partial L_i}{\partial q} \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} + \frac{\partial L_i}{\partial q_{t_i}} \delta q_{t_i} \right) \mathrm{d}t_i$$




Integration by parts (wrt $t_i$ only) yields

$$\delta \int_{\Gamma_i} L_i \,\mathrm{d}t_i = \int_{\Gamma_i} \left( \left( \frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} \right) \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} \right) \mathrm{d}t_i + \frac{\partial L_i}{\partial q_{t_i}} \delta q \bigg|_p$$

Since $p$ is an interior point of the curve, we cannot set $\delta q(p) = 0$!

Arbitrary $\delta q$ and $\delta q_{t_1}$ so we find:

## Multi-time Euler-Lagrange equations

$$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_1}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}}$$

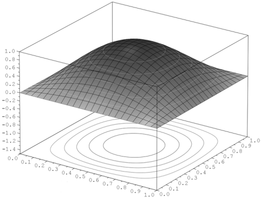
# PDEs (2-dimensional)

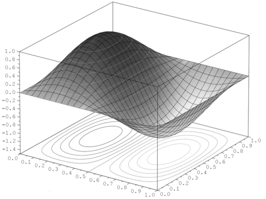
## Pluri-Lagrangian (Lagrangian multi-form) principle

Given a 2-form

$$\mathcal{L} = \sum_{i,j} L_{ij}[q] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $q(t_1, \ldots, t_N)$, such that $\int_\Gamma \mathcal{L}$ is critical on all smooth 2-dimensional surfaces $\Gamma$ in multi-time $\mathbb{R}^N$, w.r.t. variations of $q$.





Multi-time Euler-Lagrange equations are again a combination of the usual Euler-Lagrange equations and new ones involving several $L_{ij}$.

Examples: potential KdV hierarchy, AKNS hierarchy, ...

# 2-form example

Consider the first two equations of the potential KdV hierarchy:

$$q_{t_2} = q_{xxx} + 3q_x^2,$$
$$q_{t_3} = q_{xxxxx} + 10q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3,$$

where we set $t_1 = x$. These are Lagrangian, in a weak sense, with

$$L_{12} = \frac{1}{2}q_x q_{t_2} - \frac{1}{2}q_x q_{xxx} - q_x^3,$$
$$L_{13} = \frac{1}{2}q_x q_{t_3} - \frac{1}{2}q_{xxx}^2 + 5q_x q_{xx}^2 - \frac{5}{2}q_x^4.$$

A suitable coefficient $L_{23}$ of

$$\mathcal{L} = L_{12}\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}\,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

can be found (nontrivial task!), depending on

$$q_x, q_{xx}, \ldots, \qquad q_{t_2}, q_{xt_2}, \ldots, \qquad q_{t_3}, q_{xt_3}, \ldots$$

# 2-form example

The usual Euler-Lagrange equations of $L_{12}$ and $L_{13}$ yield

$$q_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x}(q_{xxx} + 3q_x^2)$$
$$q_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x}(q_{xxxxx} + 10q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3),$$

but the multi-time Euler-Lagrange equations of $\mathcal{L}$ consist of

$$q_{t_2} = q_{xxx} + 3q_x^2$$
$$q_{t_3} = q_{xxxxx} + 10q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3,$$

and consequences thereof.

# Connections to established concepts

- We can pass between the pluri-Lagrangian and Hamiltonian formalisms for 1-forms* and 2-forms†.

  The Hamiltonians are in involution if and only if $\mathrm{d}\mathcal{L} = 0$ on solutions.

- Lagrangian 2-forms can be derived from matrix Lax pairs with a rational dependence on the spectral parameter.‡

- The flows of a pluri-Lagrangian system are variational symmetries of each other if and only if $\mathrm{d}\mathcal{L} = 0$ on solutions.§

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* Suris. Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms. J. Geometric Mechanics, 2013

† V. Hamiltonian structures for integrable hierarchies of Lagrangian PDEs Open Communications in Nonlinear Mathematical Physics, 2021.

‡ Sleigh, Nijhoff, Caudrelier. A variational approach to Lax representations. Journal of Geometry and Physics, 2019.

§ Petrera, V. Variational symmetries and pluri-Lagrangian structures for integrable hierarchies of PDEs. European Journal of Mathematics, 2021

# Discretisation of Hamiltonian systems

| | | |
|---|---|---|
| Hamiltonian ODE | $\rightarrow$ | symplectic map |
| Liouville-integrable system | $\rightarrow$ | commuting symplectic maps<br>(or symplectic map with conserved quantities?) |
| Hamiltonian PDE | $\rightarrow$ | partial difference equation:<br>multisymplectic map on a lattice? |

**Variational principles are easier to discretise**

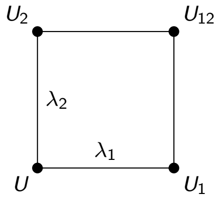
Lagrangian multiforms provide a unified perspective

# Quad equations

Discrete integrable equation for $U : \mathbb{Z}^2 \to \mathbb{R}$:

$$\mathcal{Q}(U, U_1, U_2, U_{12}, \lambda_1, \lambda_2) = 0$$

- Subscripts of $U$ denote lattice shifts.
- $\lambda_1, \lambda_2$ are lattice parameters.



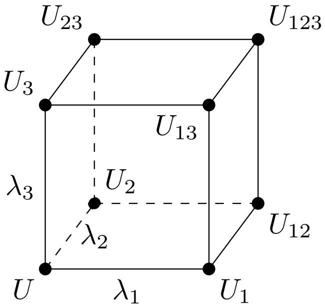

Discrete analogue of commuting flows:

**Multi-dimensional consistency**

The three ways of calculating $U_{123}$, using

$$\mathcal{Q}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j) = 0,$$

and its shifts, give the same result.




Example: lattice potential KdV:

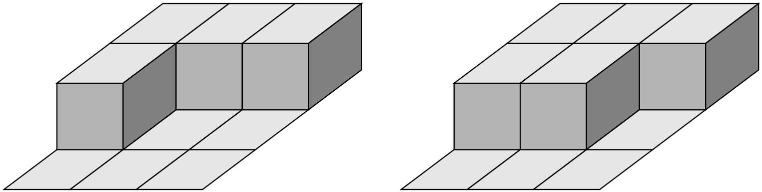
$$(U - U_{12})(U_1 - U_2) - \lambda_1 + \lambda_2 = 0$$

# Variational principle for quad equations

For some discrete 2-form

$$\mathcal{L}(\square_{ij}) = \mathcal{L}(U, U_i, U_j, U_{ij}, \lambda_i, \lambda_j),$$

the action $\sum_{\square \in \Gamma} \mathcal{L}(\square)$ is critical on all 2-surfaces $\Gamma$ in $\mathbb{Z}^N$ simultaneously.



Discretising Hamiltonian structures is ambiguous. But the discrete and continuous variational principles are essentially the same.

Lobb, Nijhoff. Lagrangian multiforms and multidimensional consistency. J. Phys. A. 2009.

# Semi-discrete Lagrangian multiforms

Consider a sequence of particles on a line:


One discrete (space) dimension, many continuous times.

Toda lattice: exponential forces between neighbours,

$$q_{t_1 t_1} = \exp(\bar{q} - q) - \exp(q - \underline{q}). \tag{T1}$$

(T1) is part of a hierarchy. Its next member is

$$q_{t_2} = q_{t_1}^2 + \exp(\bar{q} - q) + \exp(q - \underline{q}). \tag{T2}$$

It possesses a semi-discrete Lagrangian 2-form with the following coefficients ("0" stands for the discrete direction):

$$\begin{aligned} L_{01} &= \tfrac{1}{2} q_{t_1}^2 - \exp(\bar{q} - q), \\ L_{02} &= q_{t_1} q_{t_2} - \tfrac{1}{3} q_{t_1}^3 - (q_{t_1} + \bar{q}_{t_1}) \exp(\bar{q} - q), \\ L_{12} &= \tfrac{1}{4} \left( \bar{q}_{t_2} - \bar{q}_{t_1 t_1} - \bar{q}_{t_1}^2 \right)^2. \end{aligned}$$

# Semi-discrete Lagrangian multiforms

For the semi-discrete Lagrangian 2-form with coefficients

$$\begin{aligned} L_{01} &= \tfrac{1}{2} q_{t_1}^2 - \exp(\bar{q} - q), \\ L_{02} &= q_{t_1} q_{t_2} - \tfrac{1}{3} q_{t_1}^3 - (q_{t_1} + \bar{q}_{t_1}) \exp(\bar{q} - q), \\ L_{12} &= \tfrac{1}{4} \left( \bar{q}_{t_2} - \bar{q}_{t_1 t_1} - \bar{q}_{t_1}^2 \right)^2 . \end{aligned}$$

the multi-time Euler-Lagrange equations are

$$q_{t_1 t_1} = \exp(\bar{q} - q) - \exp(q - \underline{q}), \tag{T1}$$

$$q_{t_2} = q_{t_1}^2 + \exp(\bar{q} - q) + \exp(q - \underline{q}), \tag{T2}$$

and

$$\tfrac{1}{2} q_{t_2 t_2} - q_{t_1 t_1} q_{t_2} - 2 q_{t_1 t_2} q_{t_1} - \tfrac{1}{2} q_{t_1 t_1 t_1 t_1} + 3 q_{t_1}^2 q_{t_1 t_1} = 0.$$

The multiform produces a scalar PDE at a single lattice site.

It can be shown that the system (T1)–(T2) implies this PDE.

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Sleigh, V. Semi-discrete Lagrangian 2-forms and the Toda lattice. J. Phys. A. 2022.

# Summary

- Lagrangian multiforms (pluri-Lagrangian systems) provide a unified approach to various types of integrable systems:

  ODEs and PDEs,
  discrete, semi-discrete, and continuous.

- Closedness of the Lagrangian form, i.e. $\mathrm{d}\mathcal{L} = 0$, is related to variational symmetries and Hamiltonians in involution.

- To do:
  - Relation to bi-Hamiltonian structures.
  - Characterisation of special solutions.
  - Relations between integrable systems of different kinds.
  - Links to differential geometry.
  - Application to gauge theories (infinite-dimensional symmetry groups).
    $\hookrightarrow$ Noether's second theorem.
  - . . .

Thank you for your attention!