A Lagrangian perspective on integrability

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1. Introduction

Many integrable systems can be understood as hierarchies, consisting of the integrable equation and its symmetries, with a (bi-)Hamiltonian structure. Here, we present a lesser-known Lagrangian perspective on integrability.

Our approach, which is known by the names pluri-Lagrangian systems and Lagrangian multiforms, captures a whole hierarchy in a single variational principle. It can be used to describe discrete and continuous integrable systems of various dimensions. In some cases it leads to surprising relations between integrable equations of different types.

2. Liouville integrability

A Hamiltonian system with n degrees of freedom is integrable if its Hamilton function is part of a family of n independent functions $H_1, \ldots, H_n$ that satisfy $\{H_i, H_j\} = 0$.

Properties.

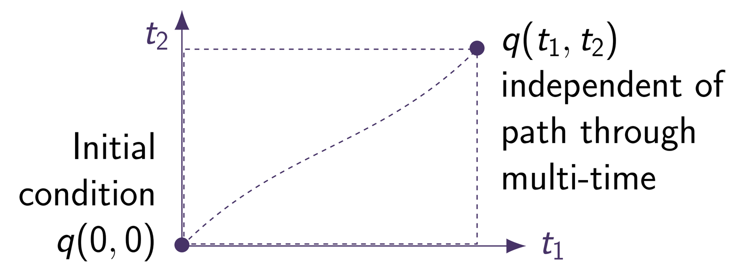
- $H_1, \ldots, H_n$ are conserved quantities.
- Each H_i can be used as a Hamilton function and defines a dynamical system.
- The flows of these systems commute with each other, so we can consider simultaneous solutions $q(t_1, \ldots, t_n)$ of $\frac{\partial q}{\partial t_i} = \{H_i, q\}$.
- Dynamics stays on a common level set of $H_1, \ldots, H_n$.
- Dynamics linear in action-angle coordinates.

Can we characterise Liouville integrability from the Lagrangian side, i.e. using a variational principle?

3. Variational principle in multi-time

Minimal example: two commuting ODEs. Associate times t_1, t_2 to the flows.

The evolution is parametrised by functions $q : \mathbb{R}^2 \to Q$ from the (t_1, t_2)-plane, called multi-time, to the configuration space Q.



(Take $\mathbb{R}^n$ instead of $\mathbb{R}^2$ to include additional commuting flows.)

Usual Lagrangian description

Infinitesimal variations of $q : \mathbb{R}^2 \to Q$ leave

$$\int L_1(q, q_{t_1})\,\mathrm{d}t_1$$

and

$$\int L_2(q, q_{t_1}, q_{t_2})\,\mathrm{d}t_2$$

invariant. (Notation: $q_{t_i} = \frac{\partial q}{\partial t_i}$.)

Pluri-Lagrangian principle

Consider the 1-form

$$\mathcal{L}[q] = L_1(q, q_{t_1})\,\mathrm{d}t_1 + L_2(q, q_{t_1}, q_{t_2})\,\mathrm{d}t_2.$$

For every curve $\gamma : [0,1] \to \mathbb{R}^2$ in multi-time and every variation v with suitable boundary conditions:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_\gamma \mathcal{L}[q + \varepsilon v]\Big|_{\varepsilon=0} = 0.$$

4. Multi-time Euler-Lagrange equations

Taking variations of q leads to the usual Euler-Lagrange equations

$$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_{t_i}} = 0 \qquad \text{(EL1)}$$

as well as

$$\frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}} \quad \text{and} \quad \frac{\partial L_i}{\partial q_{t_j}} = 0 \text{ if } i \neq j. \qquad \text{(EL2)}$$

For $\mathcal{L}$ depending on higher derivatives, natural generalisations hold.

Theorem. The following are equivalent:

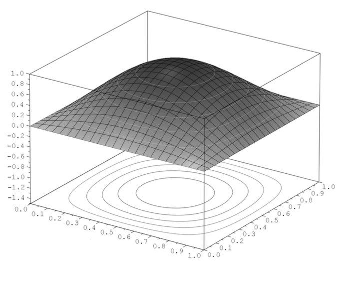
- q satisfies the pluri-Lagrangian principle.
- Equations (EL1)-(EL2) hold [3, 5]
- $\delta \mathrm{d}\mathcal{L} = 0$ [6]

Theorem. The following are equivalent:

- $\mathrm{d}\mathcal{L} = 0$ when evaluated on solutions to (EL1)-(EL2).
- Lagrangian multiform principle: on solutions to (EL1)-(EL2), the action is critical with respect to variations of the curve γ (as opposed to variations of q only). [1, 4]
- Vanishing Poisson brackets between the corresponding Hamiltonian functions. In fact we have $\{H_i, H_j\} = \partial_i \lrcorner \partial_j \lrcorner \mathrm{d}\mathcal{L} + (\text{EL equations})^2$ [9, 12]

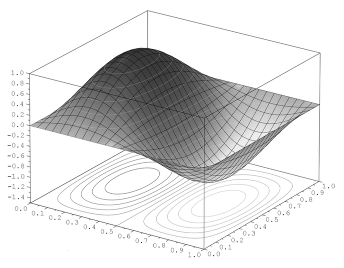
5. Partial differential equations

Hierarchies of PDEs share their space variables but have separate time variables.



For 2-dimensional PDEs, we consider a 2-form

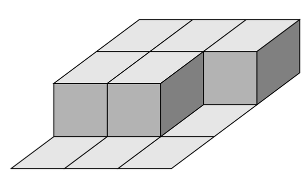
$$\mathcal{L} = \sum_{i,j} L_{ij}[q]\,\mathrm{d}t_i \wedge \mathrm{d}t_j.$$



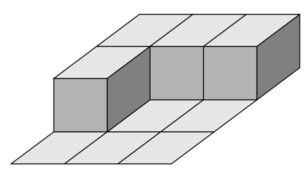
Look for $q : \mathbb{R}^n \to Q$ such that the integral of $\mathcal{L}$ is critical over every surface in multi-time.

Lagrangian 2-forms are known for potential KdV [5], AKNS [6], ... A 3-form is known for KP [7].

6. Difference equations



For difference equations on elementary squares of $\mathbb{Z}^2$, the action sum of a discrete 2-form should be critical on every discrete surface in $\mathbb{Z}^3$.



This perspective has played an important role in the study of integrable partial difference equations [1, 2, 4]. Continuum limits were studied in [8].

7. One-form example

The Kepler Lagrangian

$$L_1 = \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|}$$

can be combined with

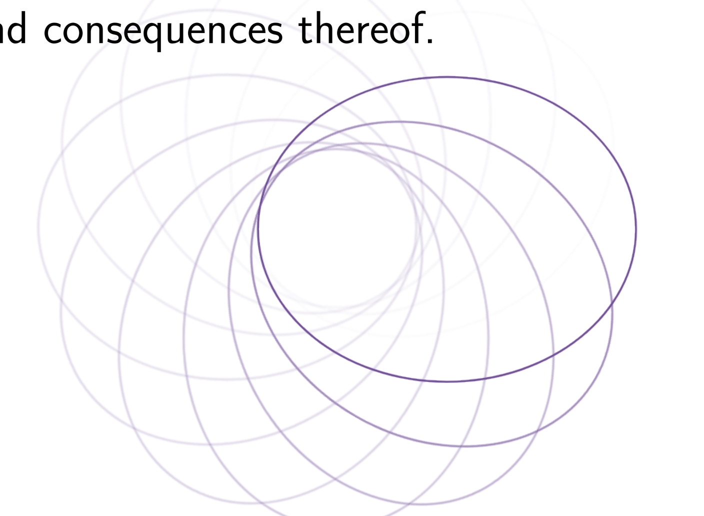
$$L_2 = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot e$$

to form a 1-form $\mathcal{L} = L_1\,\mathrm{d}t_1 + L_2\,\mathrm{d}t_2$, yielding the multi-time EL equations

$$q_{t_1 t_1} = -\frac{q}{|q|^3}, \quad \text{(inverse square law)}$$

$$q_{t_2} = e \times q \quad \text{(rotation)}$$

and consequences thereof.



8. Two-form Example

Consider the first two equations of the potential KdV hierarchy:

$$u_{t_2} = Q_2 = u_{xxx} + 3u_x^2,$$
$$u_{t_3} = Q_3 = u_{xxxxx} + 10u_x u_{xxx} + 5u_{xx}^2 + 10u_x^3,$$

where we set $t_1 = x$. These are Lagrangian, in a weak sense, with

$$L_{12} = \tfrac{1}{2}u_x u_{t_2} - \tfrac{1}{2}u_x u_{xxx} - u_x^3,$$
$$L_{13} = \tfrac{1}{2}u_x u_{t_3} - \tfrac{1}{2}u_{xxx}^2 + 5u_x u_{xx}^2 - \tfrac{5}{2}u_x^4.$$

The usual Euler-Lagrange equations of L_{12} and L_{13} yield

$$u_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x}Q_2 \quad \text{and} \quad u_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x}Q_3,$$

but with a suitable coefficient* L_{23} of

$$\mathcal{L} = L_{12}\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}\,\mathrm{d}t_2 \wedge \mathrm{d}t_3,$$

the multi-time Euler-Lagrange equations of $\mathcal{L}$ consist of

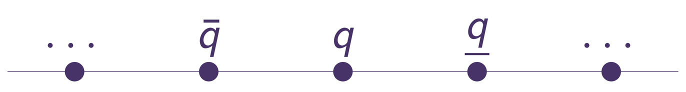
$$u_{t_2} = Q_2 \quad \text{and} \quad u_{t_3} = Q_3,$$

and consequences thereof. [5, 8]

$$\begin{aligned}
{}^* L_{23} = {} & 3u_x^5 - \tfrac{15}{2}u_x^2 u_{xx}^2 + (10u_x^3 + \tfrac{7}{2}u_{xx}^2 + 3u_x u_{xxx})u_{xxx} \\
& - (6u_x u_{xx} + \tfrac{1}{2}u_{xxxx})u_{xxxx} + (\tfrac{3}{2}u_x^2 + \tfrac{1}{2}u_{xxx})u_{xxxxx} \\
& - (5u_x^3 + \tfrac{5}{2}u_{xx}^2 + 5u_x u_{xxx} + \tfrac{1}{2}u_{xxxxx})u_{t_2} \\
& + (10u_x u_{xx} + u_{xxxx})u_{xt_2} - u_{xxx}u_{xxt_2} \\
& + (\tfrac{3}{2}u_x^2 + \tfrac{1}{2}u_{xxx})u_{t_3} - \tfrac{1}{2}u_{xx}u_{xt_3} + \tfrac{1}{2}u_x u_{xxt_3}
\end{aligned}$$

9. Semi-discrete equations

Consider a sequence of particles on a line:



(One discrete dimension, many continuous times.)

Example. The Toda lattice

$$q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q}) \qquad \text{(T1)}$$

is part of a hierarchy. Its next member is

$$q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q}), \qquad \text{(T2)}$$

where $q_i = q_{t_i}$ etc.

It possesses a semi-discrete Lagrangian 2-form with the coefficients ("0" represents discrete direction):

$$L_{01} = \tfrac{1}{2}q_1^2 - \exp(\bar{q} - q),$$
$$L_{02} = q_1 q_2 - \tfrac{1}{3}q_1^3 - (q_1 + \bar{q}_1)\exp(\bar{q} - q),$$
$$L_{12} = \tfrac{1}{4}\left(\bar{q}_{t_2} - \bar{q}_{11} - \bar{q}_1^2\right)^2.$$

The multi-time Euler-Lagrange equations are (T1)–(T2) and the following Boussinesq-type PDE:

$$\tfrac{1}{2}q_{22} - q_{11}q_2 - 2q_{12}q_1 - \tfrac{1}{2}q_{1111} + 3q_1^2 q_{11} = 0.$$

The Lagrangian multiform produces a scalar PDE at a single lattice site, in contrast to known PDEs in two variables associated to the Toda hierarchy. [10]

Similarly, a semi-discrete 2-form for the potential Volterra lattice leads to an NLS-type scalar PDE.

10. Multiforms on Lie groups

Lagrangian multiforms can describe symmetries that do not commute with each other if we replace euclidean multi-time by a Lie group.

This setup is computationally similar to the commutative case, with $\frac{\partial}{\partial t_i}$ replaced by Lie algebra elements ξ and generators ∂_ξ of their action.

The relation between Poisson brackets and $\mathrm{d}\mathcal{L}$ now reads

$$\{H_\xi, H_\nu\} = H_{[\xi,\nu]} + \partial_\xi \lrcorner \partial_\nu \lrcorner \mathrm{d}\mathcal{L} + (\text{EL eqs})^2,$$

encoding the Lie algebra structure. [11]

11. A curious connection

Consider the Lagrangian 2-form $\mathcal{L}$ on $\mathbb{R}^3$ with $L_{ij} = (c_i - c_j)\frac{u_{ij}^2}{u_i u_j}$.

Then $\delta \mathrm{d}\mathcal{L} = 0$ gives the 3d PDEs

$$(c_i - c_j)\frac{u_{ij}}{u_i u_j} + (c_j - c_k)\frac{u_{jk}}{u_j u_k} + (c_k - c_i)\frac{u_{ik}}{u_i u_k} = 0, \qquad 🕸$$

$$u_{ijk} = \frac{1}{2}\left(\frac{u_{ij}u_{ik}}{u_i} + \frac{u_{ij}u_{jk}}{u_j} + \frac{u_{ik}u_{jk}}{u_k}\right). \qquad 🕷$$

The system 🕸,🕷 is equivalent to the multi-time Euler-Lagrange equations, in particular it implies the 2d PDE

$$u_{iijj} = \frac{u_{ij}u_{ijj}}{u_j} + \frac{u_{ij}u_{iij}}{u_i} + \frac{u_{ii}u_{ijj}}{u_i} + \frac{u_{jj}u_{iij}}{u_j} - \frac{u_{ii}u_{ij}^2}{u_i^2} - \frac{u_{jj}u_{ij}^2}{u_j^2} - \frac{u_{ii}u_{ij}u_{jj}}{u_i u_j}.$$

Equation 🕸 is called the Veronese Web Equation and is related to Einstein-Weyl geometry. Equation 🕷 is related to Egorov geometry.

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