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Multi-Time Euler-Lagrange Equations and Double Zeroes

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Lagrangian Multiform Theory and Pluri-Lagrangian Systems

Hangzhou, October 2023

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Variational principle in multi-time

A simultaneous solution is a function

$$q : \mathbb{R}^N \to Q \quad \text{(multi-time to configuration space)}$$

such that $\frac{\partial q}{\partial t_1}$ generates the dynamical system and $\frac{\partial q}{\partial t_i}$ its symmetries.

Pluri-Lagrangian principle

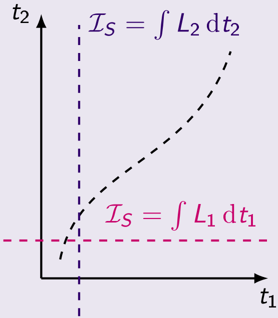
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q]\,\mathrm{d}t_i .$$

Look for dynamical variables $q(t_1, \ldots, t_N)$ such that the action

$$\mathcal{I}_S = \int_S \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve S in multi-time $\mathbb{R}^N$



Multi-time Euler-Lagrange equations

q_I: mixed partial derivative of q defined by a string $I = t_{i_1} \dots t_{i_k}$.

If I is empty then $q_I = q$.

Denote by $\dfrac{\delta_i}{\delta q_I}$ the variational derivative in the direction of t_i wrt q_I:

$$\frac{\delta_i f}{\delta q_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial f}{\partial q_{It_i^\alpha}} \qquad = \frac{\partial f}{\partial q_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial f}{\partial q_{It_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial f}{\partial q_{It_i^2}} - \dots$$

Consider $\mathcal{L}[q] = \sum_i L_i[q] \, \mathrm{d}t_i$ with $L_i[q] = L_i(q, q_{t_i}, q_{t_i t_j}, \dots)$.

Multi-time Euler-Lagrange equations / Multi-time EL eqns

Usual Euler-Lagrange equations: $\dfrac{\delta_i L_i}{\delta q_I} = 0 \qquad \forall I \not\ni t_i,$

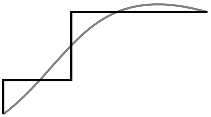
Additional conditions: $\dfrac{\delta_i L_i}{\delta q_{It_i}} = \dfrac{\delta_j L_j}{\delta q_{It_j}} \qquad \forall I,$

Derivation of the multi-time Euler-Lagrange equations

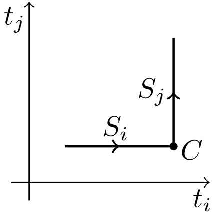
Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[q] \, \mathrm{d}t_i$.

Lemma

If the action $\int_S \mathcal{L}$ is critical on all stepped curves S in $\mathbb{R}^N$, then it is critical on all smooth curves.



Variations are local, so it is sufficient to look at an L-shaped curve $S = S_i \cup S_j$.

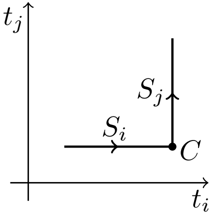


Suris, V. On the Lagrangian structure of integrable hierarchies. Advances in Discrete Differential Geometry, 2016.

Derivation of the multi-time Euler-Lagrange equations

On one of the straight pieces, S_i ($i \neq 1$), we get

$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \sum_I \frac{\partial L_i}{\partial q_I} \delta q_I \,\mathrm{d}t_i$$
$$= \int_{S_i} \sum_{I \not\ni t_i} \sum_{\alpha=0}^{\infty} \frac{\partial L_i}{\partial q_{It_i^\alpha}} \delta q_{It_i^\alpha} \,\mathrm{d}t_i$$



Integration by parts (wrt t_i only) yields

$$\delta \int_{S_i} L_i \,\mathrm{d}t_i = \int_{S_i} \sum_{I \not\ni t_i} \frac{\delta_i L_i}{\delta q_I} \delta q_I \mathrm{d}t_i + \sum_I \frac{\partial L_i}{\partial q_{It_i}} \delta q_I \bigg|_C$$

Since p is an interior point of the curve, we cannot set $\delta q(C) = 0$!

Multi-time Euler-Lagrange equations

$$\frac{\delta_i L_i}{\delta q_I} = 0 \quad \forall I \not\ni t_i, \qquad \text{and} \qquad \frac{\delta_i L_i}{\delta q_{It_i}} = \frac{\delta_j L_j}{\delta q_{It_j}}$$

Newtonian system with symmetries

Consider $\mathcal{L} = \sum_i L_i \,\mathrm{d}t_i$ with

$$\begin{aligned} L_1 &= \tfrac{1}{2}|q_1|^2 - V_i(q) \\ L_i &= q_1 \cdot q_i - H_i(q, q_1) \end{aligned}$$

Multi-time Euler-Lagrange equations

$$\frac{\delta_1 L_1}{\delta q} = 0 \quad \Rightarrow \quad \frac{\partial L_1}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L_1}{\partial q_1} = 0 \qquad \Rightarrow \quad q_{11} = -V'(q)$$

$$\frac{\delta_i L_i}{\delta q} = 0 \quad \Rightarrow \quad \frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_i} = 0 \qquad \Rightarrow \quad q_{1i} = -\frac{\partial H_i}{\partial q}$$

$$\frac{\delta_i L_i}{\delta q_1} = 0 \quad \Rightarrow \quad \frac{\partial L_i}{\partial q_1} = 0, \qquad \Rightarrow \quad q_i = \frac{\partial H_i}{\partial q_1}$$

$$\frac{\delta_i L_i}{\delta q_i} = \frac{\delta_j L_j}{\delta q_j} \quad \Rightarrow \quad \frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}} \qquad \text{trivially satisfied}$$

Exterior derivative of $\mathcal{L}$

As before, consider $L_i = q_1 \cdot q_i - H_i(q, q_1)$

Multi-time Euler-Lagrange equations:

$$q_i = \frac{\partial H_i}{\partial q_1} \qquad \text{and} \qquad q_{1i} = -\frac{\partial H_i}{\partial q}$$

Coefficient of $\mathrm{d}\mathcal{L}$

$$\frac{\mathrm{d}L_j}{\mathrm{d}t_i} - \frac{\mathrm{d}L_i}{\mathrm{d}t_j} = \left(q_{1i} + \frac{\partial H_i}{\partial q} \right) \left(q_j - \frac{\partial H_j}{\partial q_i} \right) - \left(q_{1j} + \frac{\partial H_j}{\partial q} \right) \left(q_i - \frac{\partial H_i}{\partial q_i} \right)$$
$$- \cancel{\{H_i, H_j\}}$$

Observation (also for PDEs): $\mathrm{d}\mathcal{L}$ has a "double zero" on solutions.

$\mathrm{d}\mathcal{L} = 0^2$ is key to the Lagrangian multiform approach.

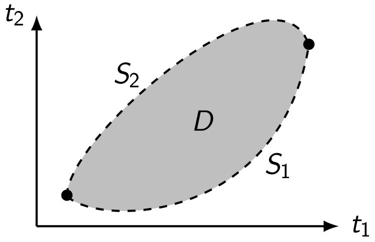
Relation between $\mathrm{d}\mathcal{L}$, double zeroes, and Poisson brackets is emphasised in:

Caudrelier, Dell'Atti, Singh. Lagrangian multiforms on coadjoint orbits for finite-dimensional integrable systems arXiv:2307.07339

Interpretation of closedness condition

If $\mathrm{d}\mathcal{L} = 0$, the action is invariant wrt variations in geometry

Deforming the curve (surface) of integration leaves action invariant.



$$\int_{S_1} \mathcal{L} - \int_{S_2} \mathcal{L} = \int_D \mathrm{d}\mathcal{L} = 0$$

Recall: pluri-Lagrangian principle only considers variations of q.

$\mathrm{d}\mathcal{L}$ provides an alternative derivation of the EL equations:

We can restrict the variational principle to simple closed curves (surfaces) of integration, i.e. boundaries of a submanifold S.

Then

$$\delta \int_{\partial S} \mathcal{L} = - \int_S \delta \mathrm{d}\mathcal{L},$$

hence the pluri-Lagrangian principle is equivalent to $\delta \mathrm{d}\mathcal{L} = 0$.

Multi-time EL equations can be obtained from variations of $\mathrm{d}\mathcal{L}$.

If $\mathrm{d}\mathcal{L}$ has a double zero, then q is critical

Multi-time Euler-Lagrange equations from $\delta \mathrm{d}\mathcal{L} = 0$

Coefficients of $\mathrm{d}\mathcal{L}$: $\quad P_{ij} = \dfrac{dL_j}{dt_i} - \dfrac{dL_i}{dt_j} \quad$ (1-form case)

$$\frac{\delta_{ij}}{\delta q_I} P_{ij} = \frac{\delta_j L_j}{\delta q_{I \setminus t_i}} - \frac{\delta_i L_i}{\delta q_{I \setminus t_j}} \quad \text{where} \quad \frac{\delta}{\delta q_{I \setminus t_i}} = \begin{cases} 0 & \text{if } I \not\ni t_i \\ \frac{\delta}{\delta q_K} & \text{if } I = K t_i \end{cases}$$

where $\quad \dfrac{\delta_{ij} L_{ij}}{\delta q_I} = \displaystyle\sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial q_{I t_i^\alpha t_j^\beta}}$

Generalises to k-forms for any k: $\quad P_{i_0 \ldots i_k} = \displaystyle\sum_{\alpha=0}^{k} (-1)^\alpha \frac{\mathrm{d}}{\mathrm{d}t_{i_\alpha}} L_{i_0 \ldots \widehat{i_\alpha} \ldots i_k}$

$$\frac{\delta_{i_0 \ldots i_k}}{\delta q_I} P_{i_0 \ldots i_k} = \sum_{\alpha=0}^{k} (-1)^\alpha \frac{\delta_{i_0 \ldots \widehat{i_\alpha} \ldots i_k}}{\delta q_{I \setminus i_\alpha}} L_{i_0 \ldots \widehat{i_\alpha} \ldots i_k}$$

Sleigh, Nijhoff, Caudrelier. Lagrangian Multiforms for the KP and the GD Hierarchy. IMRN, 2021.

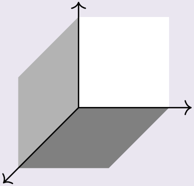
Multi-time EL equations for $\mathcal{L}[q] = \sum_{i,j} L_{ij}[q]\, \mathrm{d}t_i \wedge \mathrm{d}t_j$

For 2-forms we have three types of multi-time EL equation:

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta q_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta q_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta q_{It_k t_i}} = 0 \qquad \forall I.$$



where

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial q_{It_i^\alpha t_j^\beta}}$$

Generalisation of stepped surface approach to higher forms is possible, but trickier than using $\delta \mathrm{d}\mathcal{L} = 0$.

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Potential KdV-type hierarchy

Consider a 2-form with the first row of coefficients of the form

$$L_{1i} = \frac{1}{2} v_1 v_i - H_i(\cancel{v}, v_1, v_{11}, \ldots) \qquad (q \to v)$$

Its Euler-Lagrange equation is $\quad v_{1i} = -\dfrac{\delta_1 H_i}{\delta v} \quad \Leftarrow \quad v_i = \dfrac{\delta_1 H_i}{\delta v_1}$

We look for L_{ij} such that $\dfrac{\mathrm{d}L_{1i}}{\mathrm{d}t_j} - \dfrac{\mathrm{d}L_{1j}}{\mathrm{d}t_i} + \dfrac{\mathrm{d}L_{ij}}{\mathrm{d}t_1} = 0^2$:

$$\begin{aligned}
\frac{\mathrm{d}L_{1i}}{\mathrm{d}t_j} - \frac{\mathrm{d}L_{1j}}{\mathrm{d}t_i} &= \frac{1}{2}\left(v_{1j} + \frac{\delta_1 H_j}{\delta v} \right)\left(v_i - \frac{\delta_1 H_i}{\delta v_1} \right) - \frac{1}{2}\left(v_{1i} + \frac{\delta_1 H_i}{\delta v} \right)\left(v_j - \frac{\delta_1 H_j}{\delta v_1} \right) \\
&\quad + \frac{1}{2} \cancel{\left(\frac{\delta_1 H_j}{\delta v}\frac{\delta_1 H_i}{\delta v_1} - \frac{\delta_1 H_i}{\delta v}\frac{\delta_1 H_j}{\delta v_1} \right)} = \{H_j, H_i\} \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t_1}\Bigg(\underbrace{\frac{1}{2}\frac{\delta_1 H_i}{\delta v} v_j - \frac{1}{2}\frac{\delta_1 H_j}{\delta v} v_i + \sum_{\alpha \geq 1}\left(\frac{\delta_1 H_i}{\delta v_{1\alpha}} v_{1\alpha j} - \frac{\delta_1 H_j}{\delta v_{1\alpha}} v_{1\alpha i} \right)}_{-L_{ij}} \Bigg)
\end{aligned}$$

Potential KdV-type hierarchy

The multi-time Euler-Lagrange equations of this 2-form are

$$\frac{\delta_{1i} L_{1i}}{\delta v} = 0 \quad \Rightarrow \quad v_{1i} = -\frac{\delta_1 H_i}{\delta v}$$

$$\frac{\delta_{1i} L_{1i}}{\delta v_1} = \frac{\delta_{ji} L_{ji}}{\delta v_j} \quad \Rightarrow \quad v_i = \frac{\delta_1 H_i}{\delta v_1}$$

Formulating the construction in terms of double zeroes of $\mathrm{d}\mathcal{L}$ explains the "miracle" of finding the evolutionary equations as Euler-Lagrange equations.

Early works relied on $\mathrm{d}\mathcal{L} = 0$ but did not explicitly use the double-zero property:

Suris, V. On the Lagrangian structure of integrable hierarchies. 2016

V. Hamiltonian structures for integrable hierarchies of Lagrangian PDEs. 2021

Double zeroes did appear in works on variational symmetries and 2-forms:

Sleigh, Nijhoff, Caudrelier. Variational symmetries and Lagrangian multiforms. 2020

Petrera, V. Variational symmetries and pluri-Lagrangian structures for integrable hierarchies of PDEs. 2021

More general kinetic term

Consider $$L_{1i} = P(v, v_1, v_{11}, \ldots)v_i - H_i(v, v_1, v_{11}, \ldots)$$

It induces the Poisson bracket $\{H_i, H_j\} = \frac{\delta_1 H_i}{\delta v}(\mathrm{D}_P^* - \mathrm{D}_P)^{-1}\frac{\delta_1 H_j}{\delta v}$,

where D_P is the frechet derivative of P and D_P^* its adjoint.

We find

$$\begin{aligned}
L_{ij} = &-\sum_{\alpha=0}^{\infty}\left(\frac{\delta_1 H_i}{\delta v_{1^\alpha}} v_{1^\alpha j} - \frac{\delta_1 H_j}{\delta v_{1^\alpha}} v_{1^\alpha i}\right) - \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\alpha-1}(-1)^\beta v_{i1^{\alpha-\beta-1}}\partial_1^\beta\left(\frac{\partial P}{\partial v_{1^\alpha}} v_j\right) \\
&- \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\alpha-1}(-1)^\beta\left(\partial_1^{\alpha-\beta-1}(\mathrm{D}_P^* - \mathrm{D}_P)^{-1}\frac{\delta_1 H_i}{\delta v}\right)\partial_1^\beta\left(\frac{\partial P}{\partial v_{1^\alpha}} v_j\right) \\
&- \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\alpha-1}(-1)^\beta(\partial_1^{\alpha-\beta-1} v_j)\partial_1^\beta\left(\frac{\partial P}{\partial v_{1^\alpha}}(\mathrm{D}_P^* - \mathrm{D}_P)^{-1}\frac{\delta_1 H_i}{\delta v}\right).
\end{aligned}$$

Question. For which P and H_i is $(\mathrm{D}_P^* - \mathrm{D}_P)^{-1}\frac{\delta_1 H_i}{\delta v}$ well-defined?

Question. For which P and H_i do the flows commute?

Contents

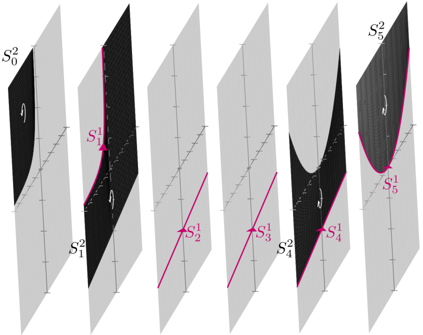
Joint work with Duncan Sleigh

Semi-discrete geometry

Only 1 discrete direction: $\mathbb{Z} \times \mathbb{R}^N$

A semi-discrete surface is a collection of surfaces and curves in $\mathbb{R}^N$ at a specified point in $\mathbb{Z}$

Intuition: curves where the surface jumps to a different value of $\mathbb{Z}$

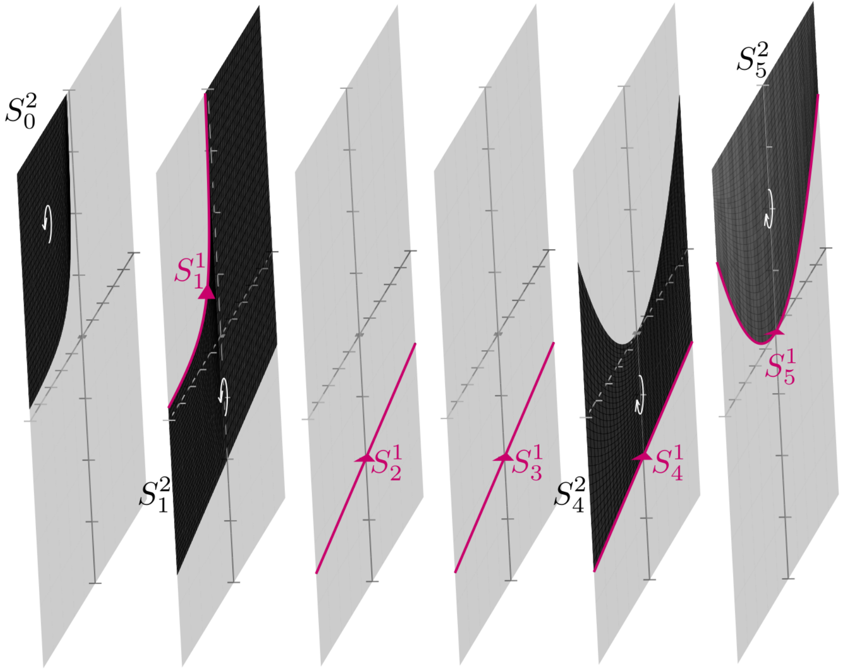


d-dimensional semi-discrete submanifold S

$$S = \left(\bigsqcup_{k \in \mathbb{Z}} S_k^{d-1} \, , \, \bigsqcup_{k \in \mathbb{Z}} S_k^d \right)$$

S_k^{d-1}: oriented $(d-1)$-dimensional submanifold of $\mathbb{R}^N$
S_k^d: oriented d-dimensional submanifold of $\mathbb{R}^N$.

Sleigh, V. Semi-discrete Lagrangian 2-forms and the Toda hierarchy. Journal of Physics A, 2022.



S_0^2
S_1^1
S_1^2
S_2^1
S_3^1
S_4^2
S_4^1
S_5^2
S_5^1

Semi-discrete geometry

- Consider (scalar) functions q of $\mathbb{Z} \times \mathbb{R}^N$.
  Superscript to emphasise lattice position: $q^{[k]} = q(k, t_1, \dots, t_N)$
- Semi-discrete d-form
  $$\mathcal{L}[q] = \left(\mathcal{L}^{d-1}[q] \, , \, \mathcal{L}^d[q] \right)$$
  consists of a $(d-1)$-form and a d-form.
- The semi-discrete integral over semi-discrete submanifold S
  $$\int_S \mathcal{L}[q] = \sum_k \int_{S_k^{d-1}} \mathcal{L}^{d-1}\left[q^{[k]} \right] + \sum_k \int_{S_k^d} \mathcal{L}^d \left[q^{[k]} \right],$$
- We have semi-discrete versions of the exterior derivative, the boundary, and Stokes theorem

Variational principle

Look for $q(k, t_1, \dots, t_N)$ such that the action $\int_S \mathcal{L}[q]$ is critical w.r.t. variations of q, simultaneously over every semi-discrete surface S.

Newtonian ($p = q_1$) lattice

Consider particles on a line:

$$\cdots \qquad \overline{q} = \mathcal{T} q = q^{[k+1]} \qquad q = q^{[k]} \qquad \underline{q} = \mathcal{T}^{-1} q = q^{[k-1]} \qquad \cdots$$

One discrete dimension, many continuous times

Suppose we have Hamiltonian densities $H_i(q, q_1, \overline{q}, \overline{q}_1, \ldots)$ such that

$$q_i = \frac{\delta_0 H_i}{\delta q_1} \qquad \text{and} \qquad q_{1\,i} = -\frac{\delta_0 H_i}{\delta q}$$

Then we consider the Lagrangians

$$L_{0\,i} = q_1 q_i - H_i$$

E.g. Toda lattice:

$$H_1 = \tfrac{1}{2} q_1^2 + \exp(\bar{q} - q)$$
$$H_2 = \tfrac{1}{3} q_1^3 + (q_1 + \bar{q}_1) \exp(\bar{q} - q)$$

Finding L_{ij}

$$
\begin{aligned}
\partial_j L_{0i} - \partial_i L_{0j} = &\left(q_{1j} + \frac{\delta_0 H_j}{\delta q} \right) \left(q_i - \frac{\delta_0 H_i}{\delta q_1} \right) - \left(q_{1i} + \frac{\delta_0 H_i}{\delta q} \right) \left(q_j - \frac{\delta_0 H_j}{\delta q_1} \right) \\
&+ \cancel{\left(\frac{\delta_0 H_i}{\delta q_1} \frac{\delta_0 H_j}{\delta q} - \frac{\delta_0 H_i}{\delta q} \frac{\delta_0 H_j}{\delta q_1} \right)} = \Delta F_{ij} \text{ if } \{\cdot, \cdot\} = 0 \\
&- \Delta \left(\sum_{\alpha=0}^{\infty} \left(\frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_j + \frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1j} \right.\right. \\
&\qquad \left.\left. - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_i - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1i} \right) \right)
\end{aligned}
$$

so we achieve a double zero of $\partial_j L_{0i} - \partial_i L_{0j} + \Delta L_{ij}$ if

$$
\begin{aligned}
L_{ij} = \sum_{\alpha=0}^{\infty} &\left(\frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_j + \frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1j} \right. \\
&\qquad \left. - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_i - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1i} \right) - \Delta F_{ij}
\end{aligned}
$$

Toda lattice

$$
\begin{aligned}
L_{01} &= \tfrac{1}{2} q_1^2 - \exp(\bar{q} - q) \\
L_{02} &= q_1 q_2 - \tfrac{1}{3} q_1^3 - (q_1 + \bar{q}_1) \exp(\bar{q} - q) \\
L_{12} &= -\exp(\bar{q} - q)^2 - \exp(\bar{q} - q)(\bar{q}_1^2 + \bar{q}_{11} - \bar{q}_2)
\end{aligned}
$$

Euler-Lagrange equations:

$$
\frac{\delta_{01} L_{01}}{\delta q} = 0 \quad \Rightarrow \quad q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q}) \tag{1}
$$

$$
\frac{\delta_{02} L_{02}}{\delta q_1} = 0 \quad \Rightarrow \quad q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q}) \tag{2}
$$

$$
\frac{\delta_{12} L_{12}}{\delta q} = 0 \quad \Rightarrow \quad \frac{1}{2} q_{22} - q_{11} q_2 - 2 q_{12} q_1 - \frac{1}{2} q_{1111} + 3 q_1^2 q_{11} = 0 \tag{3}
$$

The PDE (3) is a consequence of the DDEs (1)–(2)

because exterior derivative has a double zero on solutions of (1)–(2)

The PDE

The equation

$$\tfrac{1}{2} q_{22} - q_{11} q_2 - 2 q_{12} q_1 - \tfrac{1}{2} q_{1111} + 3 q_1^2 q_{11} = 0$$

is a Boussinesq-type integrable PDE.

(Equivalent forms have appeared in classification results.)

DDEs as Bäcklund transformations

The first two Toda equations can we written as

$$\exp(\bar{q} - q) = \frac{1}{2}(q_2 + q_{11} - q_1^2),$$
$$\exp(q - \underline{q}) = \frac{1}{2}(q_2 - q_{11} - q_1^2).$$

which give an auto-Bäcklund transformation for (3)

Mikhailov, Novikov, Wang. On classification of integrable nonevolutionary equations. Studies in Applied Mathematics, 2007.

Potential Volterra hierarchy

The equations

$$q_{11} = -q_1 \left(\bar{q}_1 - \underline{q}_1\right)$$
$$q_2 = q_1 \left(\bar{q}_1 + q_1 + \underline{q}_1\right)$$

Provide a double zero for the semi-discrete Lagrangian 2-form

$$L_{01} = q_1 \log(q_1) - q\bar{q}_1 - q_1$$
$$L_{02} = q_2 \log(q_1) - q\bar{q}_2 - \frac{1}{2}q_1^2 - q_1\bar{q}_1$$
$$L_{12} = -q_1^2\bar{q}_1 - q_1\bar{q}_1^2 + q_1\bar{q}_{11} + q_1\bar{q}_2$$

The first form of L_{12} yields the Levi system

$$2q_{11}\bar{q}_1 + 2q_1\bar{q}_{11} + 2\bar{q}_1\bar{q}_{11} - \bar{q}_{111} - \bar{q}_{12} = 0$$
$$2q_{11}\bar{q}_1 + 2q_1\bar{q}_{11} + 2q_1 q_{11} - q_{111} - q_{12} = 0$$

The second form of L_{12} yields a scalar PDE equivalent to it.

PDEs for any intergrable differential-difference equation?

Probably not. . .

- Given L_{0i}, the double-zero property provides an algorithm to find L_{ij}.
- The L_{ij} generally depend on several lattice shifts.
  Eliminating these has been done by trial and error.

Contents

Joint work with Evgeny Ferapontov

The generating PDE, stripped down

The generating PDE for KdV has a Lagrangian 2-form with coefficients

$$L_{ij} = \frac{1}{2}(\cancelto{c_i}{t_i} - \cancelto{c_j}{t_j}) \frac{u_{ij}^2}{u_i u_j} + \cancel{\frac{1}{2(t_i - t_j)} \left(n_j^2 \frac{u_i}{u_j} + n_i^2 \frac{u_j}{u_i} \right)}$$

Set $n_i = n_j = 0$ and replace t_i by constants.

Then

$$\mathrm{d}\mathcal{L} = A_{ijk} B_{ijk} \, \mathrm{d}t_i \wedge \mathrm{d}t_j \wedge \mathrm{d}t_k$$

with

$$A_{ijk} = (c_i - c_j)\frac{u_{ij}}{u_i u_j} + (c_j - c_k)\frac{u_{jk}}{u_j u_k} + (c_k - c_i)\frac{u_{ik}}{u_i u_k}$$

$$B_{ijk} = u_{ijk} - \frac{u_{ij} u_{ik}}{u_i} + \frac{u_{ij} u_{jk}}{u_j} + \frac{u_{ik} u_{jk}}{u_k}$$

Nijhoff, Hone, Joshi. On a Schwarzian PDE associated with the KdV hierarchy. Physics Letters A, 2000

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Sleigh. The Lagrangian multiform approach to integrable systems. 2021.

$$L_{ij} = \frac{1}{2}(c_i - c_j)\frac{u_{ij}^2}{u_i u_j}$$

The multi-time Euler-Lagrange equations

$$u_{iijj} - \frac{u_{ij}u_{ijj}}{u_j} - \frac{u_{iij}u_{ij}}{u_i} + \frac{u_{ii}u_{ij}^2}{u_i^2} - \frac{u_{iij}u_{ij}}{u_i} - \frac{u_{jj}u_{iij}}{u_j} + \frac{u_{jj}u_{ij}^2}{u_j^2} + \frac{u_{ii}u_{ij}u_{jj}}{u_i u_j} = 0$$

$$\frac{c_i - c_j}{u_i u_j}\left(\frac{u_{ij}^2}{2u_j} - u_{iij} + \frac{u_{ii}u_{ij}}{u_i}\right) - \frac{c_i - c_k}{u_i u_k}\left(\frac{u_{ik}^2}{2u_k} - u_{iik} + \frac{u_{ii}u_{ik}}{u_i}\right) = 0$$

$$(c_i - c_j)\frac{u_{ij}}{u_i u_j} + (c_j - c_k)\frac{u_{jk}}{u_j u_k} + (c_k - c_i)\frac{u_{ik}}{u_i u_k} = 0$$

are consequences of $A_{ijk} = 0$ and $B_{ijk} = 0$:

$$(c_i - c_j)\frac{u_{ij}}{u_i u_j} + (c_j - c_k)\frac{u_{jk}}{u_j u_k} + (c_k - c_i)\frac{u_{ik}}{u_i u_k} = 0$$

$$u_{ijk} - \frac{u_{ij}u_{ik}}{u_i} + \frac{u_{ij}u_{jk}}{u_j} + \frac{u_{ik}u_{jk}}{u_k} = 0$$

Both of these equations have geometric meaning.

$A_{ijk} = 0$

If u solves the Veronese Web Equation

$$(c_i - c_j)\frac{u_{ij}}{u_i u_j} + (c_j - c_k)\frac{u_{jk}}{u_j u_k} + (c_k - c_i)\frac{u_{ik}}{u_i u_k} = 0$$

then

$$g^{(1)} = (c_2 - c_3)^2 \frac{u_1}{u_2 u_3}\mathrm{d}t_1^2 + (c_3 - c_1)^2 \frac{u_2}{u_1 u_3}\mathrm{d}t_2^2 + (c_1 - c_2)^2 \frac{u_3}{u_1 u_2}\mathrm{d}t_3^2,$$

$$\omega = -\frac{u_{11}}{u_1}\mathrm{d}t_1 - \frac{u_{22}}{u_2}\mathrm{d}t_2 - \frac{u_{33}}{u_3}\mathrm{d}t_3$$

is an Einstein-Weyl structure, i.e. there is a torsion-free connection D s.t.

$$\mathrm{D}_i g^{(1)}_{jk} = \omega_i g^{(1)}_{jk} \qquad \text{(Weyl)}$$

$$(\mathrm{Ric}_\mathrm{D})_{(ij)} = \Lambda g_{ij} \qquad \text{(Einstein-Weyl)}$$

Calderbank, Pedersen. Einstein-Weyl geometry. Surveys in Differential Geometry, 2001.

Dunajski, Kryński. Einstein–Weyl geometry, dispersionless Hirota equation and Veronese webs. Math. Proc. Cambridge Philos. Soc, 2014

$B_{ijk} = 0$

If u solves

$$u_{ijk} - \frac{u_{ij} u_{ik}}{u_i} + \frac{u_{ij} u_{jk}}{u_j} + \frac{u_{ik} u_{jk}}{u_k} = 0$$

then the diagonal potential metric (Egorov metric)

$$g^{(2)} = u_1 \mathrm{d}t_1^2 + u_2 \mathrm{d}t_2^2 + u_3 \mathrm{d}t_3^2$$

has diagonal curvature: only nonzero components of curvature are ${R^i}_{jij}$

Lagrangian multiform connects these two geometric equations.

Do we believe in coincidence?

Dubrovin. Differential geometry of strongly integrable systems of hydrodynamic type. Functional Analysis and Its Applications, 1990.

Contents

Summary

- Different derivations of multi-time Euler-Lagrange equations (stepped curves/surfaces vs $\delta \mathrm{d}\mathcal{L}$): matter of personal taste.
- Factorisation of $\mathrm{d}\mathcal{L}$ is key to constructing examples.
- Comparing factorisation of $\mathrm{d}\mathcal{L}$ and multi-time Euler-Lagrange equations leads to nontrivial connections!
  - Differential-difference equations and PDEs.
  - PDEs and very different PDEs.

Thank you for your attention!