

# Lagrangian multiforms and symmetries of differential-difference equations

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Integrable Day

Loughborough, 24 November, 2023

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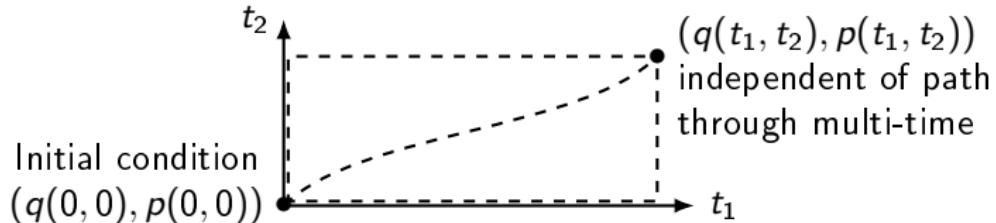
# Liouville integrability

A Hamiltonian system with Hamilton function  $H : T^*Q \cong \mathbb{R}^{2N} \rightarrow \mathbb{R}$  is **Liouville integrable** if there exist  $N$  functionally independent Hamilton functions  $H = H_1, H_2, \dots, H_N$  such that  $\{H_i, H_j\} = 0$ .

- ▶ Each  $H_i$  defines its own flow  $\phi_{H_i}^t$ :  $N$  dynamical systems
- ▶ **The flows commute:**  $\phi_{H_i}^{t_i} \circ \phi_{H_j}^{t_j} = \phi_{H_j}^{s_j} \circ \phi_{H_i}^{t_i}$ .  
(Infinitesimally:  $[X_{H_i}, X_{H_j}] = 0$ .)

We can consider  $(q, p)$  as a function of **multi-time**,  $\mathbb{R}^N \rightarrow T^*Q$ :

$$(t_1, \dots, t_N) \mapsto (q(t_1, \dots, t_N), p(t_1, \dots, t_N))$$



# Lagrangian mechanics

Lagrange function  $L : TQ \cong \mathbb{R}^{2N} \rightarrow \mathbb{R} : (q, q_t) \mapsto L(q, q_t)$

Dynamics follows curves which are minimizers (critical points) of the action

$$\int_a^b L(q, q_t) dt \quad \text{with fixed boundary values } q(a) \text{ and } q(b).$$

Minimizers satisfy the Euler-Lagrange (EL) equation  $\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial q_t} = 0$

**Proof.** Consider an arbitrary variation  $\delta q$ :

$$\delta \int_a^b L dt = \int_a^b \left( \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial q_t} \delta q_t \right) dt$$

Integration by parts yields

$$\delta \int_a^b L dt = \int_a^b \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial q_t} \right) \delta q dt + \left[ \frac{\partial L}{\partial q_t} \delta q \right]_a^b$$

EL follows because  $\delta q(a) = \delta q(b) = 0$  and  $\delta q$  is arbitrary inside  $(a, b)$ . ■

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Includes work by Nijhoff, Suris, ...

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# Lagrangian formulation of Liouville integrable system

Suppose we have Lagrange functions  $L_i$  associated to  $H_i$ . Consider

$$q : \mathbb{R}^N \rightarrow Q \quad (\text{multi-time to configuration space})$$

such that  $\frac{\partial q}{\partial t_i}$  correspond to the Hamiltonian vector fields.

## Variational (“Pluri-Lagrangian”) principle for ODEs

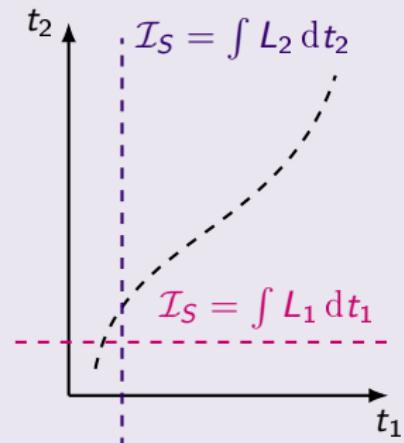
Combine the  $L_i$  into a **1-form**

$$\mathcal{L}[q] = \sum_{i=1}^N L_i[q] dt_i.$$

Look for dynamical variables  $q(t_1, \dots, t_N)$   
such that the action

$$\mathcal{I}_S = \int_S \mathcal{L}[q]$$

is critical w.r.t. **variations of  $q$** , simultaneously  
over **every curve  $S$**  in multi-time  $\mathbb{R}^N$



# Multi-time Euler-Lagrange equations

Assume that

$$L_1[q] = L_1(q, q_{t_1}),$$

$$L_i[q] = L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1$$

The multi-time Euler-Lagrange equations for  $\mathcal{L} = \sum_i L_i[q] dt_i$  are

Usual Euler-Lagrange equations:  $\frac{\partial L_i}{\partial q} - \frac{d}{dt_i} \frac{\partial L_i}{\partial q_{t_i}} = 0$

$$? : \frac{\partial L_i}{\partial q_{t_1}} = 0, \quad i \neq 1$$

Compatibility conditions:  $\frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}}$

## Example: Kepler Problem

Take

$$L_1 = \frac{1}{2} |q_{t_1}|^2 + \frac{1}{|q|}$$

$$L_2 = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot \hat{v} \quad (\hat{v} \text{ fixed unit vector})$$

In general:  $L_i = q_{t_1} q_{t_i} - H_i(q, q_{t_1})$

Multi-time Euler-Lagrange equations of  $\mathcal{L} = L_1 dt_1 + L_2 dt_2$

$$\frac{\partial L_1}{\partial q} - \frac{d}{dt_1} \frac{\partial L_1}{\partial q_{t_1}} = 0 \Rightarrow q_{t_1 t_1} = -\frac{q}{|q|^3} \quad (\text{Keplerian motion})$$

$$\frac{\partial L_2}{\partial q} - \frac{d}{dt_2} \frac{\partial L_2}{\partial q_{t_2}} = 0 \Rightarrow q_{t_1 t_2} = \hat{v} \times q_{t_1}$$

$$\frac{\partial L_2}{\partial q_{t_1}} = 0 \Rightarrow q_{t_2} = \hat{v} \times q \quad (\text{Rotation})$$

$$\frac{\partial L_1}{\partial q_{t_1}} = \frac{\partial L_2}{\partial q_{t_2}} \Rightarrow q_{t_1} = q_{t_1}$$

# Derivation of the multi-time Euler-Lagrange equations

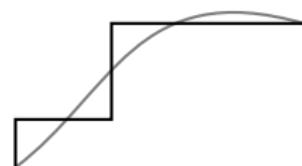
Consider a Lagrangian one-form  $\mathcal{L} = \sum_i L_i[q] dt_i$ , with

$$L_1[q] = L_1(q, q_{t_1}),$$

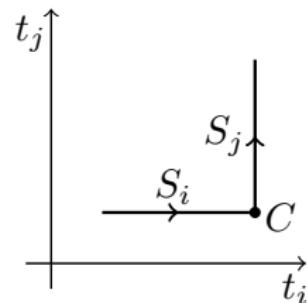
$$L_i[q] = L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1$$

## Lemma

If the action  $\int_S \mathcal{L}$  is critical on all **stepped curves**  $S$  in  $\mathbb{R}^N$ , then it is critical on all smooth curves.



Variations are local, so it is sufficient to look at an **L-shaped curve**  $S = S_i \cup S_j$ .

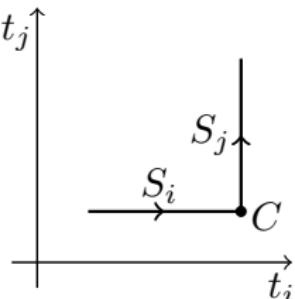


# Derivation of the multi-time Euler-Lagrange equations

On one of the straight pieces,  $S_i$  ( $i \neq 1$ ), we get

$$\delta \int_{S_i} L_i dt_i = \int_{S_i} \left( \frac{\partial L_i}{\partial q} \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} + \frac{\partial L_i}{\partial q_{t_i}} \delta q_{t_i} \right) dt_i$$

Integration by parts (wrt  $t_i$  only) yields



$$\delta \int_{S_i} L_i dt_i = \int_{S_i} \left( \left( \frac{\partial L_i}{\partial q} - \frac{d}{dt_i} \frac{\partial L_i}{\partial q_{t_i}} \right) \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} \right) dt_i + \frac{\partial L_i}{\partial q_{t_i}} \delta q \Big|_C$$

Since  $p$  is an interior point of the curve, we cannot set  $\delta q(C) = 0$ !

Arbitrary  $\delta q$  and  $\delta q_{t_1}$ , so we find:

## Multi-time Euler-Lagrange equations

$$\frac{\partial L_i}{\partial q} - \frac{d}{dt_i} \frac{\partial L_i}{\partial q_{t_i}} = 0, \quad \frac{\partial L_i}{\partial q_{t_1}} = 0, \quad \frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_i}}$$

Higher order Lagrangians  $L_i[q] = L_i(q, q_{t_i}, q_{t_i t_j}, \dots)$

For a string  $I = t_{i_1} \dots t_{i_k}$  of time variables, denote the corresponding derivative by  $q_I$ .

If  $I$  is empty then  $q_I = q$ .

Denote by  $\frac{\delta_i}{\delta q_I}$  the variational derivative in the direction of  $t_i$  wrt  $q_I$ :

$$\begin{aligned}\frac{\delta_i L_i}{\delta q_I} &= \sum_{\alpha=0}^{\infty} (-1)^{\alpha} \frac{d^{\alpha}}{dt_i^{\alpha}} \frac{\partial L_i}{\partial q_{I t_i^{\alpha}}} \\ &= \frac{\partial L_i}{\partial q_I} - \frac{d}{dt_i} \frac{\partial L_i}{\partial q_{I t_i}} + \frac{d^2}{dt_i^2} \frac{\partial L_i}{\partial q_{I t_i^2}} - \dots\end{aligned}$$

## Multi-time Euler-Lagrange equations

Usual Euler-Lagrange equations:  $\frac{\delta_i L_i}{\delta q_I} = 0 \quad \forall I \not\ni t_i,$

Additional conditions:  $\frac{\delta_i L_i}{\delta q_{I t_i}} = \frac{\delta_j L_j}{\delta q_{J t_j}} \quad \forall I,$

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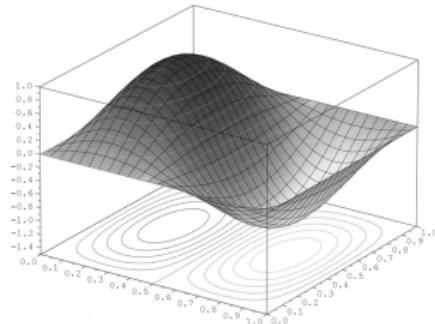
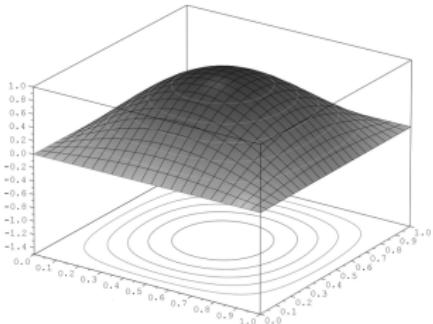
# Variational principle for PDEs ( $d = 2$ )

## Pluri-Lagrangian principle

Given a 2-form

$$\mathcal{L}[q] = \sum_{i,j} L_{ij}[q] dt_i \wedge dt_j,$$

find a field  $q : \mathbb{R}^N \rightarrow \mathbb{R}$ , such that  $\int_S \mathcal{L}[q]$  is critical on all smooth surfaces  $S$  in multi-time  $\mathbb{R}^N$ , w.r.t. variations of  $q$ .



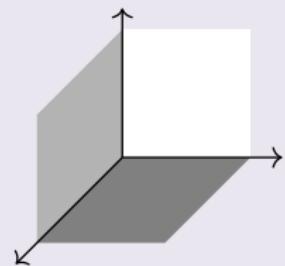
# Multi-time EL equations

for  $\mathcal{L}[q] = \sum_{i,j} L_{ij}[q] dt_i \wedge dt_j$

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = 0 \quad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta q_{It_k}} \quad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta q_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta q_{It_k t_i}} = 0 \quad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{d^\alpha}{dt_i^\alpha} \frac{d^\beta}{dt_j^\beta} \frac{\partial L_{ij}}{\partial q_{It_i^\alpha t_j^\beta}}$$

## Example: Potential KdV hierarchy

$$q_{t_2} = q_{xxx} + 3q_x^2,$$

$$q_{t_3} = q_{xxxxx} + 10q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3,$$

where we identify  $t_1 = x$ .

The differentiated equations  $q_{xt_i} = \frac{d}{dx}(\dots)$  are Lagrangian with

$$L_{12} = \frac{1}{2}q_x q_{t_2} - \frac{1}{2}q_x q_{xxx} - q_x^3,$$

$$L_{13} = \frac{1}{2}q_x q_{t_3} - \frac{1}{2}q_{xxx}^2 + 5q_x q_{xx}^2 - \frac{5}{2}q_x^4.$$

A suitable coefficient  $L_{23}$  of

$$\mathcal{L} = L_{12} dt_1 \wedge dt_2 + L_{13} dt_1 \wedge dt_3 + L_{23} dt_2 \wedge dt_3$$

can be found (nontrivial task!).

## Example: Potential KdV hierarchy

- The equations  $\frac{\delta_{12}L_{12}}{\delta q} = 0$  and  $\frac{\delta_{13}L_{13}}{\delta q} = 0$  yield

$$q_{xt_2} = \frac{d}{dx} (q_{xxx} + 3q_x^2),$$

$$q_{xt_3} = \frac{d}{dx} (q_{xxxxx} + 10q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3).$$

- The equations  $\frac{\delta_{12}L_{12}}{\delta q_x} = \frac{\delta_{32}L_{32}}{\delta q_{t_3}}$  and  $\frac{\delta_{13}L_{13}}{\delta q_x} = \frac{\delta_{23}L_{23}}{\delta q_{t_2}}$  yield

$$q_{t_2} = q_{xxx} + 3q_x^2,$$

$$q_{t_3} = q_{xxxxx} + 10q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3,$$

the evolutionary equations!

- All other multi-time EL equations are corollaries of these.

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## Exterior derivative of $\mathcal{L}$

Revisit the **Kepler problem**:  $\mathcal{L} = L_1 dt_1 + L_2 dt_2$  with

$$L_1[q] = \frac{1}{2} |q_{t_1}|^2 + \frac{1}{|q|}$$

$$L_2[q] = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot \hat{v} \quad (\hat{v} \text{ fixed unit vector})$$

Multi-time Euler-Lagrange equations:

$$q_{t_1 t_1} = -\frac{q}{|q|^3} \quad q_{t_2} = \hat{v} \times q$$

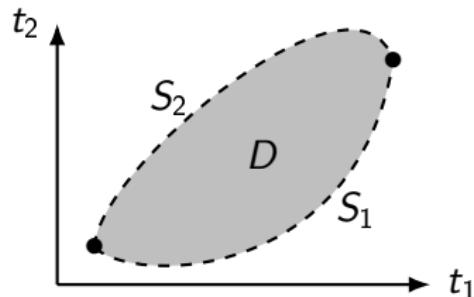
### Coefficient of $d\mathcal{L}$

$$\frac{dL_2}{dt_1} - \frac{dL_1}{dt_2} = \left( q_{t_1 t_1} + \frac{q}{|q|^3} \right) (q_{t_2} - \hat{v} \times q)$$

Observation:  $d\mathcal{L}$  typically has a “double zero” on solutions.

# Interpretation of closedness condition I

If  $d\mathcal{L} = 0$ , the action is invariant wrt variations in geometry



$$\int_{S_1} \mathcal{L} - \int_{S_2} \mathcal{L} = \int_D d\mathcal{L} = 0$$

## Lagrangian multiform principle

Require that

- ▶ pluri-Lagrangian principle holds (variations of  $q$ ),
- ▶ deforming the curve of integration leaves action invariant.

## Interpretation of closedness condition II

$d\mathcal{L}$  provides an **alternative derivation of the EL equations**:

WLOG, we can restrict the variational principle to simple closed curves, i.e. boundaries of a surface  $D$ .

Then

$$\delta \int_{\partial D} \mathcal{L} = - \int_D \delta d\mathcal{L},$$

hence the pluri-Lagrangian principle is equivalent to  $\delta d\mathcal{L} = 0$ .

If  $d\mathcal{L}$  has a **double zero** on a set of equations  $E_1 = 0, E_2 = 0, \dots$ ,

$$d\mathcal{L} = \sum_{i,j} E_i E_j dt_i \wedge dt_j$$

or

$$d\mathcal{L} = \sum_{i,j} \left( \sum_{\alpha,\beta} c_{\alpha,\beta}^{i,j} E_\alpha E_\beta \right) dt_i \wedge dt_j,$$

then  $q$  is critical if  $E_1 = 0, E_2 = 0, \dots$

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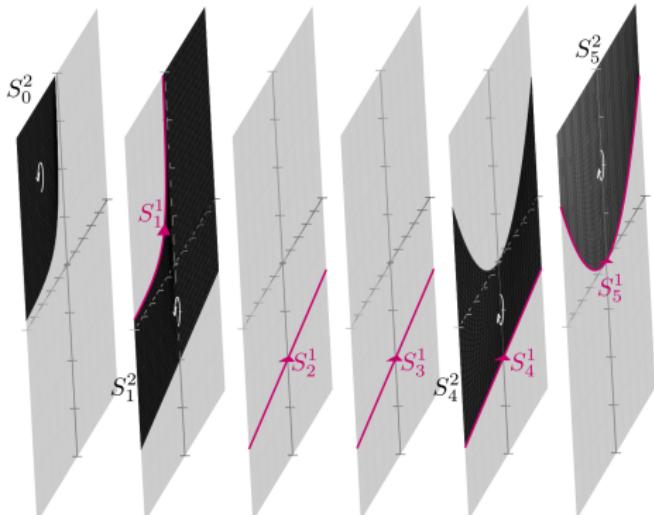
# Semi-discrete geometry

Only 1 discrete direction:

$$\mathbb{Z} \times \mathbb{R}^N$$

A **semi-discrete surface** is a collection of surfaces and curves in  $\mathbb{R}^N$  at specified points in  $\mathbb{Z}$

Intuition: curves where the surface jumps to a different value of  $\mathbb{Z}$



## Semi-discrete surface

$$S = \left( \bigsqcup_{k \in \mathbb{Z}} S_k^1, \bigsqcup_{k \in \mathbb{Z}} S_k^2 \right)$$

where  $S_k^1$  curve in  $\mathbb{R}^N$  and  $S_k^2$  is a surface  $\mathbb{R}^N$ .

## Semi-discrete geometry

- ▶ Consider (scalar) functions  $q$  of  $\mathbb{Z} \times \mathbb{R}^N$ .  
Superscript to emphasise lattice position:  $q^{[k]} = q(k, t_1, \dots, t_N)$
- ▶ Semi-discrete 2-form

$$\mathcal{L}[q] = (\mathcal{L}^1[q], \mathcal{L}^2[q])$$

consists of a 1-form and a 2-form.

- ▶ The semi-discrete integral over semi-discrete submanifold  $S$

$$\int_S \mathcal{L}[q] = \sum_k \int_{S_k^1} \mathcal{L}^1[q^{[k]}] + \sum_k \int_{S_k^2} \mathcal{L}^2[q^{[k]}],$$

- ▶ We have semi-discrete versions of the exterior derivative, the boundary, and Stokes theorem

### Variational principle

Look for  $q(k, t_1, \dots, t_N)$  such that the action  $\int_S \mathcal{L}[q]$  is critical w.r.t. variations of  $q$ , simultaneously over every semi-discrete surface  $S$ .

# Semi-discrete variational derivatives

Traditional discrete EL eqn:

$$\sum_{n \in \mathbb{Z}} \mathcal{T}^{-n} \frac{\partial L}{\partial q^{[n]}} = 0$$

Traditional semi-discrete EL eqn:

$$\sum_{n \in \mathbb{Z}} \mathcal{T}^{-n} \frac{\delta_i L}{\delta q^{[n]}} = 0$$

## Variational derivatives

$$\frac{\delta_0 L}{\delta q_I} := \frac{\partial}{\partial q_I} \sum_{n \in \mathbb{N}} \mathcal{T}^{-n} L,$$

$$\frac{\delta_{0i} L}{\delta q_I} := \frac{\delta_i}{\delta q_I} \sum_{n \in \mathbb{N}} \mathcal{T}^{-n} L.$$

Denote  $\bar{q} = \mathcal{T}q$  and  $\underline{q} = \mathcal{T}^{-1}q$ .

Examples:

$$\frac{\delta_{0i} q_{t_i}^2}{\delta q} = \frac{\delta_i q_{t_i}^2}{\delta q} = -2 \frac{d}{dt_i} q_{t_i} = -2 q_{t_i t_i},$$

$$\frac{\delta_{0i} \bar{q}_{t_i}^2}{\delta q} = \frac{\delta_i q_{t_i}^2}{\delta q} = -2 \frac{d}{dt_i} q_{t_i} = -2 q_{t_i t_i},$$

$$\frac{\delta_{0i} \underline{q}_{t_i}^2}{\delta q} = 0,$$

# Semi-discrete multi-time Euler-Lagrange equations

A field is critical if and only if the following multi-time Euler-Lagrange equations hold for all  $n \in \mathbb{Z}$ :

$$\frac{\delta_{ij} L_{ij}}{\delta q_I^{[n]}} = 0 \quad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_j}^{[n]}} - \frac{\delta_{ik} L_{ik}}{\delta q_{It_k}^{[n]}} = 0 \quad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_i t_j}^{[n]}} + \frac{\delta_{jk} L_{jk}}{\delta q_{It_j t_k}^{[n]}} + \frac{\delta_{ki} L_{ki}}{\delta q_{It_k t_i}^{[n]}} = 0 \quad \forall I,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_j}^{[n]}} + \frac{\delta_{0i} L_{0i}}{\delta q_I^{[n]}} = 0 \quad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{It_i t_j}^{[n]}} - \frac{\delta_{0j} L_{0j}}{\delta q_{It_j}^{[n]}} + \frac{\delta_{0i} L_{0i}}{\delta q_{It_i}^{[n]}} = 0 \quad \forall I,$$

If  $n$  is such that  $L_{ij}$  does not depend on  $q_I^{[n]}$  for any  $I$ , then

$$\frac{\delta_{0i} L_{0i}}{\delta q_I^{[n]}} = 0 \quad \forall I \not\ni t_i,$$

$$\frac{\delta_{0j} L_{0j}}{\delta q_{It_j}^{[n]}} - \frac{\delta_{0i} L_{0i}}{\delta q_{It_i}^{[n]}} = 0 \quad \forall I.$$

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## Newtonian ( $p = q_1$ ) lattice

Consider particles on a line:



A horizontal line with black dots representing particles. Ellipses (...) are placed before the first and after the last dot. Below the line, three expressions are aligned under each dot:

$$\begin{aligned} \dots & \\ \bar{q} &= T q & q &= q^{[k]} & \underline{q} &= T^{-1} q \\ & & & & & \\ & & = q^{[k+1]} & & & = q^{[k-1]} \end{aligned}$$

One discrete dimension, many continuous times

Suppose we have **Hamiltonian densities**  $H_i(q, q_1, \bar{q}, \bar{q}_1, \dots)$  such that

$$q_i = \frac{\delta_0 H_i}{\delta q_1} \quad \text{and} \quad q_{1i} = -\frac{\delta_0 H_i}{\delta q}$$

Then we consider the **Lagrangians**

$$L_{0i} = q_1 q_i - H_i$$

E.g. Toda lattice:  $H_1 = \frac{1}{2} q_1^2 + \exp(\bar{q} - q)$

$$H_2 = \frac{1}{3} q_1^3 + (q_1 + \bar{q}_1) \exp(\bar{q} - q)$$

# Finding $L_{ij}$

Want  $\partial_j L_{0i} - \partial_i L_{0j} + \Delta L_{ij}$  to have a double zero:

$$\begin{aligned}\partial_j L_{0i} - \partial_i L_{0j} &= \left( q_{1j} + \frac{\delta_0 H_j}{\delta q} \right) \left( q_i - \frac{\delta_0 H_i}{\delta q_1} \right) - \left( q_{1i} + \frac{\delta_0 H_i}{\delta q} \right) \left( q_j - \frac{\delta_0 H_j}{\delta q_1} \right) \\ &\quad + \left( \frac{\delta_0 H_i}{\delta q_1} \frac{\delta_0 H_j}{\delta q} - \frac{\delta_0 H_i}{\delta q} \frac{\delta_0 H_j}{\delta q_1} \right) = \Delta F_{ij} \text{ if } \{\cdot, \cdot\} = 0 \\ &\quad - \Delta \left( \sum_{\alpha=0}^{\infty} \left( \frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_j + \frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1j} \right. \right. \\ &\quad \left. \left. - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_i - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1i} \right) \right)\end{aligned}$$

Take

$$\begin{aligned}L_{ij} &= \sum_{\alpha=0}^{\infty} \left( \frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_j + \frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1j} \right. \\ &\quad \left. - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_i - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1i} \right) - F_{ij}\end{aligned}$$

## Toda lattice

$$L_{01} = \frac{1}{2}q_1^2 - \exp(\bar{q} - q)$$

$$L_{02} = q_1 q_2 - \frac{1}{3}q_1^3 - (q_1 + \bar{q}_1) \exp(\bar{q} - q)$$

$$L_{12} = -\exp(\bar{q} - q)^2 - \exp(\bar{q} - q)(\bar{q}_1^2 + \bar{q}_{11} - \bar{q}_2)$$

Euler-Lagrange equations:

$$\frac{\delta_{01} L_{01}}{\delta q} = 0 \quad \Rightarrow \quad q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q}) \quad (1)$$

$$\frac{\delta_{02} L_{02}}{\delta q_1} = 0 \quad \Rightarrow \quad q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q}) \quad (2)$$

$$\frac{\delta_{12} \tilde{L}_{12}}{\delta q} = 0 \quad \Rightarrow \quad \frac{1}{2}q_{22} - q_{11}q_2 - 2q_{12}q_1 - \frac{1}{2}q_{1111} + 3q_1^2q_{11} = 0 \quad (3)$$

The PDE (3) is a consequence of the DDEs (1)–(2)

# The PDE

The equation

$$\frac{1}{2}q_{22} - q_{11}q_2 - 2q_{12}q_1 - \frac{1}{2}q_{1111} + 3q_1^2q_{11} = 0$$

is a Boussinesq-type integrable PDE.

(Equivalent forms have appeared in the literature\*. )

## DDEs as Bäcklund transformations

The first two Toda equations can we written as

$$\exp(\bar{q} - q) = \frac{1}{2}(q_2 + q_{11} - q_1^2),$$

$$\exp(q - \underline{q}) = \frac{1}{2}(q_2 - q_{11} - q_1^2).$$

which give an auto-Bäcklund transformation for (3)

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\* see e.g. [Mikhailov, Novikov, Wang. On classification of integrable nonevolutionary equations. Studies in Applied Mathematics, 2007]

## $t_3$ -flow

At the next level we find

$$\frac{\delta_{13} L_{13}}{\delta q} = 0 \quad \rightarrow \quad q_1^3 - 3q_1 q_{11} + 6q_1 \exp(\bar{q} - q) + q_{111} - q_3 = 0,$$

which can be simplified to

$$q_3 = -2q_1^3 + 3q_1 q_2 + q_{111}.$$

(Integrable together with  $\frac{1}{2}q_{22} - q_{11}q_2 - 2q_{12}q_1 - \frac{1}{2}q_{1111} + 3q_1^2 q_{11} = 0$   
but not on its own.)

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- Other examples

### 3 Summary

# Potential Volterra hierarchy

The Volterra lattice

$$a_1 = a(\bar{a} - \underline{a})$$

Made Lagrangian by switching to the potential variable  $q$  s.t.  $a = q_1$ .

The potential Volterra equations

$$q_{11} = -q_1(\bar{q}_1 - \underline{q}_1)$$

$$q_2 = q_1(\bar{q}_1 + q_1 + \underline{q}_1)$$

Provide a double zero for the semi-discrete Lagrangian 2-form

$$L_{01} = q_1 \log(q_1) - q\bar{q}_1 - q_1$$

$$L_{02} = q_2 \log(q_1) - q\bar{q}_2 - \frac{1}{2}q_1^2 - q_1\bar{q}_1$$

$$L_{12} = -q_1^2\bar{q}_1 - q_1\bar{q}_1^2 + q_1\bar{q}_{11} + q_1\bar{q}_2$$

## Potential Volterra hierarchy

$L_{12} = -q_1^2 \bar{q}_1 - q_1 \bar{q}_1^2 + q_1 \bar{q}_{11} + q_1 \bar{q}_2$  yields the Levi system\*

$$2q_{11}\bar{q}_1 + 2q_1\bar{q}_{11} + 2\bar{q}_1\bar{q}_{11} - \bar{q}_{111} - \bar{q}_{12} = 0$$

$$2q_{11}\bar{q}_1 + 2q_1\bar{q}_{11} + 2q_1q_{11} - q_{111} - q_{12} = 0$$

We can split off a double zero:

$$\left. \begin{aligned} L_{12} &= \frac{1}{4}\bar{q}_1^3 - \frac{1}{2}\bar{q}_1\bar{q}_{11} - \frac{1}{2}\bar{q}_1\bar{q}_2 + \frac{\bar{q}_{11}^2 + 2\bar{q}_{11}\bar{q}_2 + \bar{q}_2^2}{4\bar{q}_1} \\ &\quad - \frac{(\bar{q}_2 + \bar{q}_{11} - 2q_1\bar{q}_1 - \bar{q}_1^2)^2}{4\bar{q}_1} \end{aligned} \right\} = \tilde{L}_{12}$$

$\tilde{L}_{12}$  yields a scalar PDE equivalent to the Levi system:

$$-\frac{3}{2}q_1^2 q_{11} + q_1 q_{12} + \frac{q_{11}^3}{2q_1^2} - \frac{q_{11}q_{111}}{q_1} + \frac{1}{2}q_{1111} + \frac{q_{12}q_2}{q_1} - \frac{q_{11}q_2^2}{2q_1^2} - \frac{1}{2}q_{22} = 0$$

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\* Levi. Nonlinear differential difference equations as Backlund transformations. J Phys A, 1981

# Scalar PDE for any intergrable DDE?

Probably not...

- ! Inverse problem of the calculus of variations.
- ✓ Given  $L_{0i}$ , the double-zero property provides an algorithm to find  $L_{ij}$ .
- ! The  $L_{ij}$  generally depend on several lattice shifts.

Related literature:

- ▶ [Adler, Shabat. *On a class of Toda chains*. Theoretical and Mathematical Physics, 1997] contains both examples presented.  
They obtain *non-scalar* PDEs.
- ▶ [Yamilov. *Symmetries as integrability criteria for differential difference equations*. J Phys A, 2006] provides a list of candidate DDEs.  
Recent observations by P Xenitidis\* on Lagrangian multiforms for master symmetries make it feasible to look for multiforms for this list.

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\*Presented at the BIRS-IAMS workshop on “Lagrangian Multiform Theory and Pluri-Lagrangian Systems”

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# Summary

- ▶ The **Lagrangian multiform** (or pluri-Lagrangian) principle describes symmetries and integrability.  
Applies to ODEs and PDEs, discrete and continuous.
- ▶ **Semi-discrete** Lagrangian 2-forms are recent and surprising rich development.
- ▶ Multi-time Euler-Lagrange equations related to “double zeros” of  $d\mathcal{L}$ .  
These two perspectives allow us to relate PDEs to DDEs.
- ▶ Most of this story is presented in  
[Duncan Sleigh, MV. **Semi-discrete Lagrangian 2-forms and the Toda hierarchy**. J Phys A, 2022]

Thank you for your attention!