A variational principle for integrable systems

Mats Vermeeren

Milano

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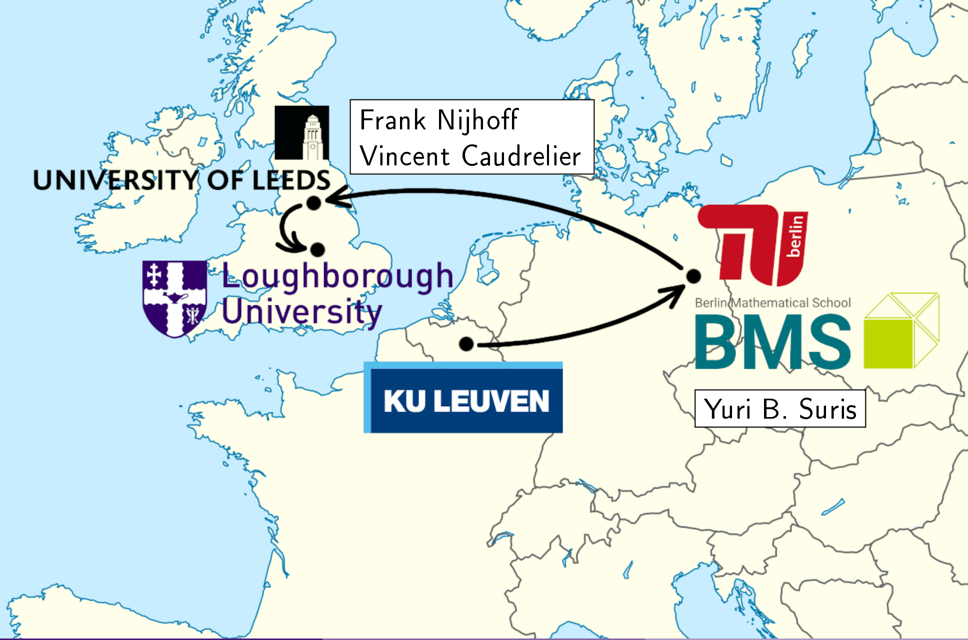
About me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Loughborough University
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berlin
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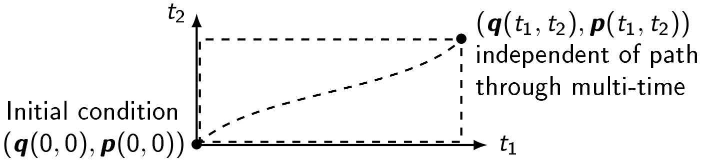
Liouville integrability

A Hamiltonian system with Hamilton function $H : T^*Q \cong \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \dots H_N$ such that $\{H_i, H_j\} = 0$.

- Each H_i defines its own flow $\phi^t_{H_j}$: N dynamical systems
- The flows commute: $\phi^{t_i}_{H_i} \circ \phi^{t_j}_{H_j} = \phi^{s_j}_{H_j} \circ \phi^{t_i}_{H_i}$.
  (Infinitesimally: $[X_{H_i}, X_{H_j}] = 0$.)

We can consider $(\boldsymbol{q}, \boldsymbol{p})$ as a function of multi-time, $\mathbb{R}^N \to T^*Q$:

$$(t_1, \dots, t_N) \mapsto (\boldsymbol{q}(t_1, \dots, t_N), \boldsymbol{p}(t_1, \dots, t_N))$$



Lagrangian mechanics

Lagrange function $L : TQ \cong \mathbb{R}^{2N} \to \mathbb{R} : (\boldsymbol{q}, \boldsymbol{q}_t) \mapsto L(\boldsymbol{q}, \boldsymbol{q}_t)$

Dynamics follows curves which are minimizers (critical points) of the action

$$\int_a^b L(\boldsymbol{q}, \boldsymbol{q}_t)\,\mathrm{d}t \qquad \text{with fixed boundary values } \boldsymbol{q}(a) \text{ and } \boldsymbol{q}(b).$$

Minimizers satisfy the Euler-Lagrange (EL) equation $\quad \dfrac{\partial L}{\partial \boldsymbol{q}} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \boldsymbol{q}_t} = 0$

Proof. Consider an arbitrary variation $\delta\boldsymbol{q}$:

$$\delta \int_a^b L\,\mathrm{d}t = \int_a^b \left(\frac{\partial L}{\partial \boldsymbol{q}} \delta\boldsymbol{q} + \frac{\partial L}{\partial \boldsymbol{q}_t} \delta\boldsymbol{q}_t \right) \mathrm{d}t$$

Integration by parts yields

$$\delta \int_a^b L\,\mathrm{d}t = \int_a^b \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \boldsymbol{q}_t} \right) \delta\boldsymbol{q}\,\mathrm{d}t + \left[\frac{\partial L}{\partial \boldsymbol{q}_t} \delta\boldsymbol{q} \right]_a^b$$

EL follows because $\delta\boldsymbol{q}(a) = \delta\boldsymbol{q}(b) = 0$ and $\delta\boldsymbol{q}$ is arbitrary inside (a, b). ∎

Lagrangian formulation of Liouville integrable system

Suppose we have Lagrange functions L_i associated to H_i. Consider

$$\boldsymbol{q} : \mathbb{R}^N \to Q \quad \text{(multi-time to configuration space)}$$

Variational ("Pluri-Lagrangian") principle for ODEs

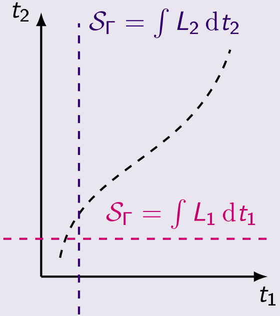
Combine the L_i into a 1-form

$$\mathcal{L}[\boldsymbol{q}] = \sum_{i=1}^{N} L_i[\boldsymbol{q}] \, \mathrm{d}t_i .$$

Look for $\boldsymbol{q}(t_1, \ldots, t_N)$ such that the action

$$\mathcal{S}_\Gamma = \int_\Gamma \mathcal{L}[\boldsymbol{q}]$$

is critical w.r.t. variations of $\boldsymbol{q}$, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$



Lagrangian multiform principle: the action is the same for all curves.

Multi-time Euler-Lagrange equations

Assume that

$$L_1[\boldsymbol{q}] = L_1(\boldsymbol{q}, \boldsymbol{q}_{t_1}),$$
$$L_i[\boldsymbol{q}] = L_i(\boldsymbol{q}, \boldsymbol{q}_{t_1}, \boldsymbol{q}_{t_i}), \quad i \neq 1$$

The multi-time Euler-Lagrange equations for $\mathcal{L} = \sum_i L_i[\boldsymbol{q}] \, \mathrm{d}t_i$ are

Usual Euler-Lagrange equations: $\dfrac{\partial L_i}{\partial \boldsymbol{q}} - \dfrac{\mathrm{d}}{\mathrm{d}t_i} \dfrac{\partial L_i}{\partial \boldsymbol{q}_{t_i}} = 0$

? : $\dfrac{\partial L_i}{\partial \boldsymbol{q}_{t_1}} = 0, \quad i \neq 1$

Compatibility conditions: $\dfrac{\partial L_i}{\partial \boldsymbol{q}_{t_i}} = \dfrac{\partial L_j}{\partial \boldsymbol{q}_{t_j}}$

Example: Kepler Problem

Take

$$L_1 = \frac{1}{2}|\boldsymbol{q}_{t_1}|^2 + \frac{1}{|\boldsymbol{q}|}$$

$$L_2 = \boldsymbol{q}_{t_1} \cdot \boldsymbol{q}_{t_2} + (\boldsymbol{q}_{t_1} \times \boldsymbol{q}) \cdot \hat{\boldsymbol{v}} \qquad (\hat{\boldsymbol{v}} \text{ fixed unit vector})$$

In general: $L_i = \boldsymbol{q}_{t_1} \cdot \boldsymbol{q}_{t_i} - H_i(\boldsymbol{q}, \boldsymbol{q}_{t_1})$

Multi-time Euler-Lagrange equations of $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$

$$\frac{\partial L_1}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L_1}{\partial \boldsymbol{q}_{t_1}} = 0 \quad \Rightarrow \quad \boldsymbol{q}_{t_1 t_1} = -\frac{\boldsymbol{q}}{|\boldsymbol{q}|^3} \qquad \text{(Keplerian motion)}$$

$$\frac{\partial L_2}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t_2}\frac{\partial L_2}{\partial \boldsymbol{q}_{t_2}} = 0 \quad \Rightarrow \quad \boldsymbol{q}_{t_1 t_2} = \hat{\boldsymbol{v}} \times \boldsymbol{q}_{t_1}$$

$$\frac{\partial L_2}{\partial \boldsymbol{q}_{t_1}} = 0 \quad \Rightarrow \quad \boldsymbol{q}_{t_2} = \hat{\boldsymbol{v}} \times \boldsymbol{q} \qquad \text{(Rotation)}$$

$$\frac{\partial L_1}{\partial \boldsymbol{q}_{t_1}} = \frac{\partial L_2}{\partial \boldsymbol{q}_{t_2}} \quad \Rightarrow \quad \boldsymbol{q}_{t_1} = \boldsymbol{q}_{t_1}$$

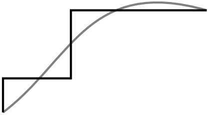
Derivation of the multi-time Euler-Lagrange equations

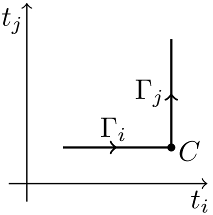
Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[\boldsymbol{q}] \, \mathrm{d}t_i$, with

$$L_1[\boldsymbol{q}] = L_1(\boldsymbol{q}, \boldsymbol{q}_{t_1}),$$
$$L_i[\boldsymbol{q}] = L_i(\boldsymbol{q}, \boldsymbol{q}_{t_1}, \boldsymbol{q}_{t_i}), \quad i \neq 1$$

Lemma

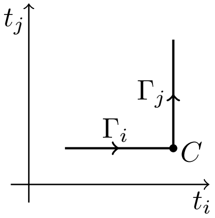
If the action $\int_\Gamma \mathcal{L}$ is critical on all stepped curves Γ in $\mathbb{R}^N$, then it is critical on all smooth curves.





Variations are local, so it is sufficient to look at an L-shaped curve $\Gamma = \Gamma_i \cup \Gamma_j$.

Derivation of the multi-time Euler-Lagrange equations



On one of the straight pieces, Γ_i ($i \neq 1$), we get

$$\delta \int_{\Gamma_i} L_i \,\mathrm{d}t_i = \int_{\Gamma_i} \left(\frac{\partial L_i}{\partial \boldsymbol{q}} \delta \boldsymbol{q} + \frac{\partial L_i}{\partial \boldsymbol{q}_{t_1}} \delta \boldsymbol{q}_{t_1} + \frac{\partial L_i}{\partial \boldsymbol{q}_{t_i}} \delta \boldsymbol{q}_{t_i} \right) \mathrm{d}t_i$$

Integration by parts (wrt t_i only) yields

$$\delta \int_{\Gamma_i} L_i \,\mathrm{d}t_i = \int_{\Gamma_i} \left(\left(\frac{\partial L_i}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial \boldsymbol{q}_{t_i}} \right) \delta \boldsymbol{q} + \frac{\partial L_i}{\partial \boldsymbol{q}_{t_1}} \delta \boldsymbol{q}_{t_1} \right) \mathrm{d}t_i + \frac{\partial L_i}{\partial \boldsymbol{q}_{t_i}} \delta \boldsymbol{q} \bigg|_C$$

Since C is an interior point of the curve, we cannot set $\delta \boldsymbol{q}(C) = 0$!

Arbitrary $\delta \boldsymbol{q}$ and $\delta \boldsymbol{q}_{t_1}$, so we find:

Multi-time Euler-Lagrange equations

$$\frac{\partial L_i}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial \boldsymbol{q}_{t_i}} = 0, \qquad \frac{\partial L_i}{\partial \boldsymbol{q}_{t_1}} = 0, \qquad \frac{\partial L_i}{\partial \boldsymbol{q}_{t_i}} = \frac{\partial L_j}{\partial \boldsymbol{q}_{t_j}}$$

Higher order Lagranigans $L_i[\boldsymbol{q}] = L_i(\boldsymbol{q}, \boldsymbol{q}_{t_i}, \boldsymbol{q}_{t_i t_j}, \ldots)$

For a string $I = t_{i_1} \ldots t_{i_k}$ of time variables, denote the corresponding derivative by $\boldsymbol{q}_I$.

If I is empty then $\boldsymbol{q}_I = \boldsymbol{q}$.

Denote by $\dfrac{\delta_i}{\delta \boldsymbol{q}_I}$ the variational derivative in the direction of t_i wrt $\boldsymbol{q}_I$:

$$\begin{aligned}
\frac{\delta_i L_i}{\delta q_I} &= \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial L_i}{\partial \boldsymbol{q}_{I t_i^\alpha}} \\
&= \frac{\partial L_i}{\partial \boldsymbol{q}_I} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial \boldsymbol{q}_{I t_i}} + \frac{\mathrm{d}^2}{\mathrm{d}t_i^2} \frac{\partial L_i}{\partial \boldsymbol{q}_{I t_i^2}} - \ldots
\end{aligned}$$

Multi-time Euler-Lagrange equations

Usual Euler-Lagrange equations: $\dfrac{\delta_i L_i}{\delta \boldsymbol{q}_I} = 0 \qquad \forall I \not\ni t_i,$

Additional conditions: $\dfrac{\delta_i L_i}{\delta \boldsymbol{q}_{I t_i}} = \dfrac{\delta_j L_j}{\delta \boldsymbol{q}_{I t_j}} \qquad \forall I,$

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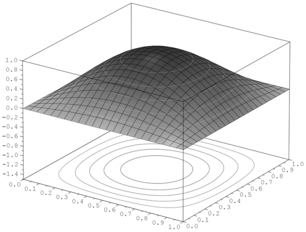
Variational principle for PDEs ($d = 2$)

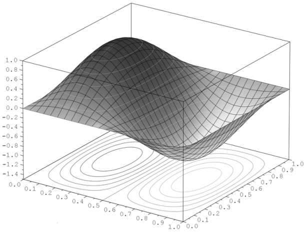
Pluri-Lagrangian principle

Given a 2-form

$$\mathcal{L}[q] = \sum_{i,j} L_{ij}[q] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

find a field $q : \mathbb{R}^N \to \mathbb{R}$, such that $\int_\Gamma \mathcal{L}[q]$ is critical on all smooth surfaces Γ in multi-time $\mathbb{R}^N$, w.r.t. variations of q.





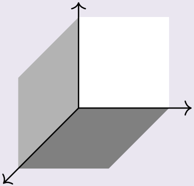
Multi-time EL equations

for $\mathcal{L}[q] = \sum_{i,j} L_{ij}[q] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{I t_j}} = \frac{\delta_{ik} L_{ik}}{\delta q_{I t_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{I t_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta q_{I t_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta q_{I t_k t_i}} = 0 \qquad \forall I.$$



Where

$$\frac{\delta_{ij} L_{ij}}{\delta q_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial q_{I t_i^\alpha t_j^\beta}}$$

Example: Potential KdV hierarchy

$q_{t_2} = q_{xxx} + 3q_x^2,$
$q_{t_3} = q_{xxxxx} + 10q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3,$
where we identify $t_1 = x$.

The differentiated equations $q_{xt_i} = \frac{\mathrm{d}}{\mathrm{d}x}(\cdots)$ are Lagrangian with

$$L_{12} = \frac{1}{2}q_x q_{t_2} - \frac{1}{2}q_x q_{xxx} - q_x^3,$$
$$L_{13} = \frac{1}{2}q_x q_{t_3} - \frac{1}{2}q_{xxx}^2 + 5q_x q_{xx}^2 - \frac{5}{2}q_x^4.$$

A suitable coefficient L_{23} of

$$\mathcal{L} = L_{12}\,\mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}\,\mathrm{d}t_2 \wedge \mathrm{d}t_3$$

can be found (nontrivial task!).

Example: Potential KdV hierarchy

- The equations $\frac{\delta_{12} L_{12}}{\delta q} = 0$ and $\frac{\delta_{13} L_{13}}{\delta q} = 0$ yield

$$q_{xt_2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(q_{xxx} + 3q_x^2\right),$$
$$q_{xt_3} = \frac{\mathrm{d}}{\mathrm{d}x}\left(q_{xxxxx} + 10q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3\right).$$

- The equations $\frac{\delta_{12} L_{12}}{\delta q_x} = \frac{\delta_{32} L_{32}}{\delta q_{t_3}}$ and $\frac{\delta_{13} L_{13}}{\delta q_x} = \frac{\delta_{23} L_{23}}{\delta q_{t_2}}$ yield

$$q_{t_2} = q_{xxx} + 3q_x^2,$$
$$q_{t_3} = q_{xxxxx} + 10q_x q_{xxx} + 5q_{xx}^2 + 10q_x^3,$$

  the evolutionary equations!

- All other multi-time EL equations are corollaries of these.

Hamiltonian structure

Set $p[q] = \frac{\delta_1 L_{1j}}{\delta q_j}$ (independent of j), then $L_{1j} = p[q] q_j - h_j[q]$

$$0 = \frac{\delta_1 L_{1j}}{\delta q} = \underbrace{\sum_k \left((-1)^k \partial_x^k \frac{\partial p[q]}{\partial q_{x^k}} - \frac{\partial p[q]}{\partial q_{x^k}} \partial_x^k \right)}_{\mathcal{E}_p} q_j - \frac{\delta h_j}{\delta q}$$

Then, $q_j = \mathcal{E}_p^{-1} \frac{\delta h_j}{\delta q}$, so h_j are Hamiltonians wrt the Poisson bracket

$$\{ \textstyle\int f \, \mathrm{d}x, \int g \, \mathrm{d}x \} = - \displaystyle\int \frac{\delta f}{\delta q} \mathcal{E}_p^{-1} \frac{\delta g}{\delta q} \, \mathrm{d}x$$

Example: potential KdV

$$p[q] = \frac{1}{2} q_x, \qquad \mathcal{E}_p = -\partial_x, \qquad \{ \textstyle\int f \, \mathrm{d}x, \int g \, \mathrm{d}x \} = \displaystyle\int \frac{\delta f}{\delta q} \partial_x^{-1} \frac{\delta g}{\delta q} \, \mathrm{d}x$$

In the KdV variable $u = q_x$, this becomes

$$\{ \textstyle\int f \, \mathrm{d}x, \int g \, \mathrm{d}x \} = \displaystyle\int \left(\partial_x \frac{\delta f}{\delta u} \right) \frac{\delta g}{\delta u} \, \mathrm{d}x$$

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Exterior derivative of $\mathcal{L}$

Revisit the Kepler problem: $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$ with

$$L_1[\boldsymbol{q}] = \frac{1}{2}|\boldsymbol{q}_{t_1}|^2 + \frac{1}{|\boldsymbol{q}|}$$

$$L_2[\boldsymbol{q}] = \boldsymbol{q}_{t_1} \cdot \boldsymbol{q}_{t_2} + (\boldsymbol{q}_{t_1} \times \boldsymbol{q}) \cdot \hat{\boldsymbol{v}} \qquad (\hat{\boldsymbol{v}} \text{ fixed unit vector})$$

Multi-time Euler-Lagrange equations:

$$\boldsymbol{q}_{t_1 t_1} = -\frac{\boldsymbol{q}}{|\boldsymbol{q}|^3} \qquad\qquad \boldsymbol{q}_{t_2} = \hat{\boldsymbol{v}} \times \boldsymbol{q}$$

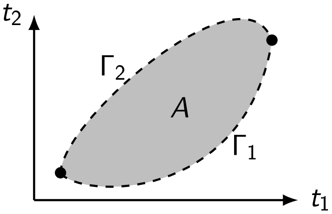
Coefficient of $\mathrm{d}\mathcal{L}$

$$\frac{\mathrm{d}L_2}{\mathrm{d}t_1} - \frac{\mathrm{d}L_1}{\mathrm{d}t_2} = \left(\boldsymbol{q}_{t_1 t_1} + \frac{\boldsymbol{q}}{|\boldsymbol{q}|^3} \right) \cdot (\boldsymbol{q}_{t_2} - \hat{\boldsymbol{v}} \times \boldsymbol{q})$$

Observation: $\mathrm{d}\mathcal{L}$ typically has a “double zero” on solutions.

Interpretation of closedness condition I

If $\mathrm{d}\mathcal{L} = 0$, the action is invariant wrt variations in geometry



$$\int_{\Gamma_1} \mathcal{L} - \int_{\Gamma_2} \mathcal{L} = \int_A \mathrm{d}\mathcal{L} = 0$$

Lagrangian multiform principle

Require that

- pluri-Lagrangian principle holds (variations of $\boldsymbol{q}$),
- deforming the curve of integration leaves action invariant.

Interpretation of closedness condition II

$\mathrm{d}\mathcal{L}$ provides an alternative derivation of the EL equations:

WLOG, we can restrict the variational principle to simple closed curves, i.e. boundaries of a surface D.

Then

$$\delta \int_{\partial D} \mathcal{L} = - \int_D \delta \mathrm{d}\mathcal{L},$$

hence the pluri-Lagrangian principle is equivalent to $\delta \mathrm{d}\mathcal{L} = 0$.

If $\mathrm{d}\mathcal{L}$ has a double zero on a set of equations $E_1 = 0$, $E_2 = 0$, ...,

$$\mathrm{d}\mathcal{L} = \sum_{i,j} E_i E_j \, \mathrm{d}t_i \wedge \mathrm{d}t_j$$

or

$$\mathrm{d}\mathcal{L} = \sum_{i,j} \left(\sum_{\alpha,\beta} c^{i,j}_{\alpha,\beta} E_\alpha E_\beta \right) \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

then $\boldsymbol{q}$ is critical if $E_1 = 0$, $E_2 = 0$, ...

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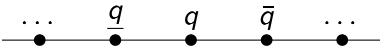
Semi-discrete systems

Consider particles on a line: 1 discrete dimension, many continuous times



Denote $\quad q_1 = q_{t_1} = \frac{\mathrm{d}q}{\mathrm{d}t_1}, \quad q_{11} = q_{t_1 t_1} = \frac{\mathrm{d}^2 q}{\mathrm{d}t_1^2}, \quad$ etc.

Toda lattice: exponential nearest-neighbour interaction

$$q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q}).$$

Part of a hierarchy. First symmetry:

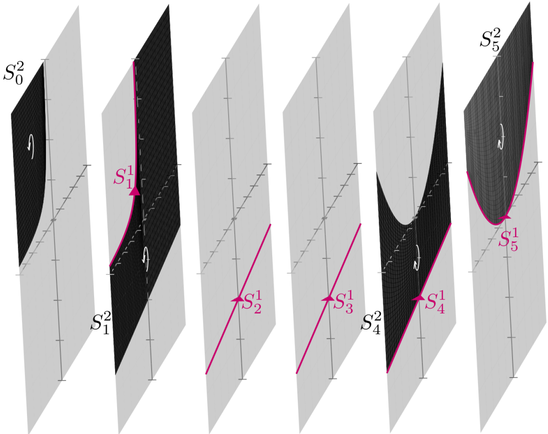
$$q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q})$$

Semi-discrete geometry

Consider the case with only 1 discrete direction: $\mathbb{Z} \times \mathbb{R}^N$

A semi-discrete surface is a collection of surfaces and curves in $\mathbb{R}^N$, each at a specified point in $\mathbb{Z}$

Intuition: curves where the surface jumps to a different value of $\mathbb{Z}$



Semi-discrete geometry

- Consider (scalar) functions q of $\mathbb{Z} \times \mathbb{R}^N$.
  Superscript to emphasise lattice position: $q^{[k]} = q(k, t_1, \ldots, t_N)$
- Semi-discrete 2-form $\mathcal{L}[q]$ is part 1-form and part 2-form:
  components L_{0j} are integrated over curves,
  components L_{ij} integrated over surfaces.
- We have semi-discrete versions of the exterior derivative, the boundary, and Stokes theorem

Variational principle

Look for $q(k, t_1, \ldots, t_N)$ such that the action

$$\int_\Gamma \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every semi-discrete surface Γ.

Toda lattice

Lagrangians ("0" for discrete direction)

$$L_{01} = \frac{1}{2} q_1^2 - \exp(\bar{q} - q)$$

$$L_{02} = q_1 q_2 - \frac{1}{3} q_1^3 - (q_1 + \bar{q}_1) \exp(\bar{q} - q)$$

$$L_{12} = -\frac{1}{4} \left(q_2 - q_{11} - q_1^2 \right)^2$$

Euler-Lagrange equations:

$$\frac{\delta_{01} L_{01}}{\delta q} = 0 \qquad \to \qquad q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q})$$

$$\frac{\delta_{02} L_{02}}{\delta q_1} = 0 \qquad \to \qquad q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q})$$

$$\frac{\delta_{12} L_{12}}{\delta q} = 0 \qquad \to \qquad \frac{1}{2} q_{22} - q_{11} q_2 - 2 q_{12} q_1 - \frac{1}{2} q_{1111} + 3 q_1^2 q_{11} = 0$$

Lagrangian formalism produces a non-trivial PDE at a single lattice site.

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Non-commuting flows

What if symmetries that do not commute with each other?

$$\begin{aligned} \text{euclidean multi-time} \quad &\rightarrow \quad \text{Lie group} \\ \frac{\partial}{\partial t_i} \quad &\rightarrow \quad \begin{cases} \text{Lie algebra element } \xi \\ \text{infinitesimal generator } \partial_\xi \text{ of its action} \end{cases} \end{aligned}$$

The relation between Poisson brackets and $\mathrm{d}\mathcal{L}$ becomes

$$\{H_\xi, H_\nu\} = H_{[\xi,\nu]} + \partial_\xi \lrcorner \partial_\nu \lrcorner \mathrm{d}\mathcal{L} + (\text{EL eqs})^2,$$

where H_ξ is the Hamiltonian of ∂_ξ

Hence $\mathrm{d}\mathcal{L} = 0$ encodes the Lie algebra structure.

Example: rotation group for central force system

Multi-time: $\mathbb{R} \times SO(3)$, generators $\xi_0 (= \frac{\partial}{\partial t}), \xi_1, \xi_2, \xi_3$ with

$$[\xi_1, \xi_2] = \xi_3 \qquad \text{(and cyclic permutations of 1,2,3)}$$

Lagrangian 1-form defined by

$$L_0 = \partial_{\xi_0} \lrcorner \mathcal{L} = \frac{1}{2} |\boldsymbol{q}_0|^2 - V(|\boldsymbol{q}|)$$
$$L_i = \partial_{\xi_1} \lrcorner \mathcal{L} = \boldsymbol{q}_0 \cdot \boldsymbol{q}_i - (\boldsymbol{q} \times \boldsymbol{q}_0) \cdot \boldsymbol{e}_i \qquad \text{for } i = 1, 2, 3$$

Multi-time Euler-Lagrange equations:

$$\boldsymbol{q}_{00} = -\boldsymbol{\nabla} V(|\boldsymbol{q}|) \qquad \boldsymbol{q}_i = \boldsymbol{e}_i \times \boldsymbol{q}$$

Closure relation:

$$\begin{aligned} \partial_{\xi_i} \lrcorner \partial_{\xi_j} \lrcorner \mathrm{d}\mathcal{L} &= \partial_{\xi_i} L_j - \partial_{\xi_j} L_i - [\partial_{\xi_i}, \partial_{\xi_j}] \lrcorner \mathcal{L} \\ &= (\boldsymbol{q}_{0i} - \boldsymbol{e}_i \times \boldsymbol{q}_0) \cdot (\boldsymbol{q}_j - \boldsymbol{e}_j \times \boldsymbol{q}) \\ &\quad - (\boldsymbol{q}_{0j} - \boldsymbol{e}_j \times \boldsymbol{q}_0) \cdot (\boldsymbol{q}_i - \boldsymbol{e}_i \times \boldsymbol{q}) = 0 \end{aligned}$$

Is multi-time always a Lie group?

If

$$\{H_i, H_j\} = \sum_k C_{ij}^k H_k,$$

then yes:

$$[X_{H_i}, X_{H_j}] = \sum_k C_{ij}^k H_k$$

Example: Kepler problem with angular momentum $\boldsymbol{\ell}$ and Runge-Lenz vector $\boldsymbol{A}$. We have $\{H, \cdot\} = 0$ and

$$\{\ell_i, \ell_j\} = -\epsilon_{ijk}\ell_k\,, \qquad \{A_i, \ell_j\} = -\epsilon_{ijk}A_k\,, \qquad \{A_i, A_j\} = 2\epsilon_{ijk}H\ell_k$$

can be linearised by setting $\widetilde{A} = \frac{A}{\sqrt{-2H}}$:

$$\{\ell_i, \ell_j\} = -\epsilon_{ijk}\ell_k\,, \qquad \{\widetilde{A}_i, \ell_j\} = -\epsilon_{ijk}\widetilde{A}_k\,, \qquad \{\widetilde{A}_i, \widetilde{A}_j\} = -\epsilon_{ijk}\ell_k$$

When linearisation is not possible, we think multi-time should be a Lie groupoid. (Details have not been worked out!)

Contents

Summary

Lagrangian multiform (or pluri-Lagrangian) provides a unified perspective on integrable* ODEs and PDEs, discrete, semi-discrete and continuous.

*or Lagrangian systems with any amount of symmetry

Much work to do:

- Geometry of multi-time for general superintegrable systems?
- Multiforms as a tool for constructing solutions?
- Full development of semi-discrete case?
- Semi-discrete multiforms in geometric numerical integration?
- Applications to gauge theory?
- Application to quantum integrable systems, path integrals, . . . ?

Selected references

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