Lagrangian multiforms and conservation laws

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Joint work with Evgeny Ferapontov

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Lagrangian multiforms: variational principle for a system with its symmetries

Lagrangian 1-forms $\rightarrow$ systems of equations in 1 independent variable each

Discrete: integrable maps

Continuous: integrable ODEs / Poisson-commuting Hamiltonian systems

[Yoo-Kong, Lobb, Nijhoff, 2011] [Suris 2013]

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Lagrangian 2-forms $\rightarrow$ systems of equations in 2 independent variables each

Discrete: partial difference equations (ABS list)

[Lobb, Nijhoff, 2009] [Bobenko, Suris, 2010]

Semi-discrete: eg Toda lattice

[Sleigh, Vermeeren, 2022]

Continuous: hierarchies of (1+1)-dimensional PDEs (eg KdV)

[Suris 2016] [Suris, Vermeeren 2016]

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Lagrangian 3-forms $\rightarrow$ . . .

This picture is too rigid!

Contents

Variational principle for 2-forms

Pluri-Lagrangian principle

Given a 2-form

$$\mathcal{L}[u] = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

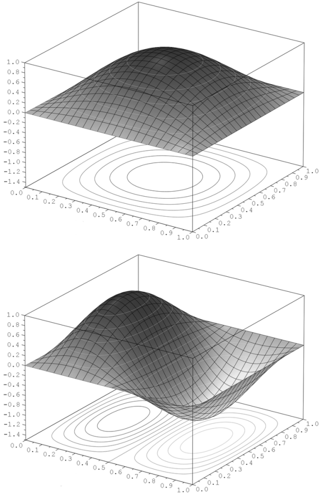
find a field $u : \mathbb{R}^N \to \mathbb{R} : (t_1, \dots t_N) \mapsto u(t_1, \dots t_N)$, such that

$$\int_\Gamma \mathcal{L}[u]$$

is critical on all smooth surfaces Γ in multi-time $\mathbb{R}^N$, w.r.t. variations of u.

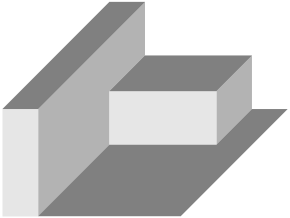
Lagrangian multiform principle

Additionally require that $\mathrm{d}\mathcal{L}[u] = 0$ on solutions: action is critical w.r.t. deformations of the surface.



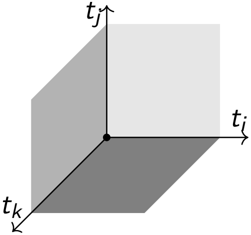
Multiform EL equations

A smooth surface can be approximated arbitrarily well by stepped surfaces.



It is sufficient to require criticality on stepped surfaces.

Variations can be taken locally, so it is sufficient to consider elementary corners.



Multiform EL equations for $\mathcal{L}[u] = \sum_{i,j} L_{ij}[u]\,\mathrm{d}t_i \wedge \mathrm{d}t_j$,

If L_{ij} depends on derivatives wrt t_i, t_j up to second order

$$(1) \qquad \frac{\delta_{ij} L_{ij}}{\delta u} = 0,$$

$$(2) \qquad \frac{\delta_{ij} L_{ij}}{\delta u_{t_j}} - \frac{\delta_{ik} L_{ik}}{\delta u_{t_k}} = 0,$$

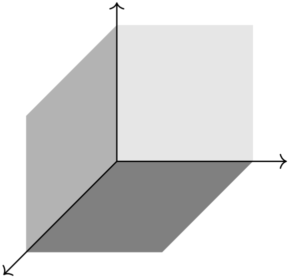
$$(3) \qquad \frac{\delta_{ij} L_{ij}}{\delta u_{t_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{t_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{t_k t_i}} = 0,$$

where

$$\frac{\delta_{ij}}{\delta u} = \frac{\partial}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial}{\partial u_{t_i}} - \frac{\mathrm{d}}{\mathrm{d}t_j}\frac{\partial}{\partial u_{t_j}} + \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\mathrm{d}}{\mathrm{d}t_j}\frac{\partial}{\partial u_{t_i t_j}}$$

$$\frac{\delta_{ij}}{\delta u_{t_j}} = \frac{\partial}{\partial u_{t_j}} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial}{\partial u_{t_i t_j}}$$

(No summation over repeated indices)



Multiform EL equations for $\mathcal{L}[u] = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$,

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where

$$\frac{\delta_{ij}}{\delta u} = \frac{\partial}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial}{\partial u_{t_i}} - \frac{\mathrm{d}}{\mathrm{d}t_j} \frac{\partial}{\partial u_{t_j}} + \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\mathrm{d}}{\mathrm{d}t_j} \frac{\partial}{\partial u_{t_i t_j}}$$

$$\frac{\delta_{ij}}{\delta u_{t_j}} = \frac{\partial}{\partial u_{t_j}} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial}{\partial u_{t_i t_j}}$$

If L_{ij} depends on u and derivatives of any order and direction.

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I,$$

where I denotes a multi-index (a combination of derivatives) and

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial u_{It_i^\alpha t_j^\beta}}$$

Exterior derivative

If the surface of integration is the boundary of a volume, $\Gamma = \partial B$, then

$$\int_\Gamma \mathcal{L} = \int_B \mathrm{d}\mathcal{L}.$$

So a necessary (and in fact sufficient) condition for criticality is that infinitesimal variations of $\mathrm{d}\mathcal{L}$ vanish:

$$\delta \mathrm{d}\mathcal{L} = 0.$$

More precisely:

$$\frac{\partial}{\partial u_I} \mathrm{d}\mathcal{L} = 0 \qquad \forall I. \qquad\qquad (*)$$

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More precisely:

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Double-zero property

If the coefficients of $\mathrm{d}\mathcal{L}$ factorise,

$$\mathrm{d}\mathcal{L} = \sum_{i<j<k} A_{ijk} B_{ijk} \, \mathrm{d}t^i \wedge \mathrm{d}t^j \wedge \mathrm{d}t^k,$$

then on solutions of the system

$$A_{ijk} = 0, \qquad B_{ijk} = 0,$$

condition $(*)$ is satisfied.

Hence the system $A_{ijk} = 0$, $B_{ijk} = 0$ implies the multiform Euler-Lagrange equations

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Variational symmetries → Lagrangian multiforms

Consider a Lagrangian L_{12} with variational symmetry $u_{t_3} = Q[u]$.

This means that $\int L_{12}\, \mathrm{d}t_1 \wedge \mathrm{d}t_2$ is invariant:

$$\int \underbrace{\mathrm{D}_{L_{12}}(Q)}_{\text{Fréchet derivative}} \mathrm{d}t_1 \wedge \mathrm{d}t_2 = \int \frac{\delta L_{12}}{\delta u} Q \, \mathrm{d}t_1 \wedge \mathrm{d}t_2 = 0 \qquad \Leftrightarrow \qquad \frac{\delta L_{12}}{\delta u} Q = \frac{\mathrm{d}M}{\mathrm{d}t_2} + \frac{\mathrm{d}N}{\mathrm{d}t_1}$$

[D Sleigh, F Nijhoff, V Caudrelier. Letters in Mathematical Physics. 2020]

Variational symmetries → Lagrangian multiforms

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Combine this with

$$\int \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} \mathrm{d}t_1 \wedge \mathrm{d}t_2 = \int \frac{\delta L_{12}}{\delta u} u_{t_3} \, \mathrm{d}t_1 \wedge \mathrm{d}t_2 \qquad \Leftrightarrow \qquad \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} = \frac{\delta L_{12}}{\delta u} u_{t_3} + \frac{\mathrm{d}\hat{M}}{\mathrm{d}t_2} + \frac{\mathrm{d}\hat{N}}{\mathrm{d}t_1}$$

to get $$\frac{\mathrm{d}L_{23}}{\mathrm{d}t_1} - \frac{\mathrm{d}L_{13}}{\mathrm{d}t_2} + \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} = \underbrace{\frac{\delta L_{12}}{\delta u}}_{A} \underbrace{(u_{t_3} - Q)}_{B}$$

So the system $A = 0$, $B = 0$ implies the multi-form Euler-Lagrange equations for $\mathcal{L} = L_{12}\, \mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13}\, \mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23}\, \mathrm{d}t_2 \wedge \mathrm{d}t_3$.

[D Sleigh, F Nijhoff, V Caudrelier. Letters in Mathematical Physics. 2020]

Example: potential KdV hierarchy – weak multiform

The potential KdV equation $u_2 = u_{111} + 3u_1^2$ has a (weak) Lagrangian $(u_k = u_{t_k})$

$$L_{12} = \frac{1}{2} u_1 u_2 - \frac{1}{2} u_1 u_{111} - u_1^3$$

and a hierarchy of variational symmetries, starting with

$$u_3 = Q[u] := u_{11111} + 10 u_1 u_{111} + 5 u_{11}^2 + 10 u_1^3.$$

This yields L_{13} and L_{23} such that

$$\frac{\mathrm{d}L_{23}}{\mathrm{d}t_1} - \frac{\mathrm{d}L_{13}}{\mathrm{d}t_2} + \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} = \frac{\delta L_{12}}{\delta u}(u_3 - Q)$$

Double zero property: multi-time EL equations follow from

$$\frac{\delta L_{12}}{\delta u} = 0 \quad \Leftrightarrow \quad u_{12} = \frac{\mathrm{d}}{\mathrm{d}t_1}(u_{111} + 3u_1^2),$$

$$u_3 = u_{11111} + 10 u_1 u_{111} + 5 u_{11}^2 + 10 u_1^3.$$

Example: potential KdV hierarchy – weak multiform

$$L_{12} = \tfrac{1}{2}u_1 u_2 - \tfrac{1}{2}u_1 u_{111} - u_1^3$$

$$L_{13} = \tfrac{1}{2}u_1 u_3 - \tfrac{5}{2}u_1^4 - 5u_1 u_{11}^2 - 5u_1^2 u_{111} - \tfrac{1}{2}u_{111}^2$$

$$L_{23} = -12u_1^5 - 15u_1^2 u_{11}^2 - 10u_1^3 u_{111} + u_{11}^2 u_{111} - 2u_1 u_{111}^2 - u_1 u_{11} u_{1111} - 5u_1^2 u_{112} + 3u_1^2 u_3 - \tfrac{1}{2}u_{1111}^2 - u_{111}u_{112} + \tfrac{1}{2}u_1 u_{113} + u_{1111}u_{12} - \tfrac{1}{2}u_{11}u_{13} + u_{111}u_3 - \tfrac{1}{2}u_2 u_3$$

Check that the factors of $\mathrm{d}\mathcal{L}$ occur as multiform Euler-Lagrange equations:

The equations $\dfrac{\delta_{12}L_{12}}{\delta u} = 0$ and $\dfrac{\delta_{13}L_{13}}{\delta u} = 0$ yield

$$u_{12} = \frac{\mathrm{d}}{\mathrm{d}t_1}\left(u_{111} + 3u_1^2\right) \quad \text{and} \quad u_{13} = \frac{\mathrm{d}}{\mathrm{d}t_1}\left(u_{11111} + 10u_1 u_{111} + 5u_{11}^2 + 10u_1^3\right).$$

Equation $\dfrac{\delta_{13}L_{13}}{\delta u_1} = \dfrac{\delta_{23}L_{23}}{\delta u_2}$ yields $\quad u_{t_3} = u_{11111} + 10u_1 u_{111} + 5u_{11}^2 + 10u_1^3.$

The equations $\dfrac{\delta_{12}L_{12}}{\delta u_2} = \dfrac{\delta_{13}L_{13}}{\delta u_3}$ and $\dfrac{\delta_{12}L_{12}}{\delta u_1} = \dfrac{\delta_{32}L_{32}}{\delta u_3}$ are trivial

Example: potential KdV hierarchy – better multiform

Previous multiform satisfies $\dfrac{\mathrm{d}L_{23}}{\mathrm{d}t_1} - \dfrac{\mathrm{d}L_{13}}{\mathrm{d}t_2} + \dfrac{\mathrm{d}L_{12}}{\mathrm{d}t_3} = \dfrac{\delta L_{12}}{\delta u}(u_3 - Q)$. Up to divergence:

$$\begin{aligned}\frac{\delta L_{12}}{\delta u}\cdot(u_3 - Q) &= \frac{\mathrm{d}}{\mathrm{d}t_1}(-u_2 + u_{111} + 3u_1^2)\cdot(u_3 - Q)\\ &\sim \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t_1}(-u_2 + u_{111} + 3u_1^2)\cdot(u_3 - Q) - \frac{1}{2}(-u_2 + u_{111} + 3u_1^2)\cdot\frac{\mathrm{d}}{\mathrm{d}t_1}(u_3 - Q)\end{aligned}$$

Replace L_{23} with $\widetilde{L}_{23} = L_{23} - \frac{1}{2}(-u_2 + u_{111} + 3u_1^2)_1(u_3 - Q)$, then

$$\frac{\mathrm{d}\widetilde{L}_{23}}{\mathrm{d}t_1} - \frac{\mathrm{d}L_{13}}{\mathrm{d}t_2} + \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t_1}(-u_2 + u_{111} + 3u_1^2)\cdot(u_3 - Q) - \frac{1}{2}(-u_2 + u_{111} + 3u_1^2)\cdot\frac{\mathrm{d}}{\mathrm{d}t_1}(u_3 - Q)$$

Example: potential KdV hierarchy – better multiform

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$$\frac{\delta L_{12}}{\delta u} \cdot (u_3 - Q) = \frac{\mathrm{d}}{\mathrm{d}t_1}(-u_2 + u_{111} + 3u_1^2) \cdot (u_3 - Q)$$
$$\sim \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t_1}(-u_2 + u_{111} + 3u_1^2) \cdot (u_3 - Q) - \frac{1}{2}(-u_2 + u_{111} + 3u_1^2) \cdot \frac{\mathrm{d}}{\mathrm{d}t_1}(u_3 - Q)$$

Replace L_{23} with $\widetilde{L}_{23} = L_{23} - \frac{1}{2}(-u_2 + u_{111} + 3u_1^2)_1(u_3 - Q)$, then

$$\frac{\mathrm{d}\widetilde{L}_{23}}{\mathrm{d}t_1} - \frac{\mathrm{d}L_{13}}{\mathrm{d}t_2} + \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t_1}(-u_2 + u_{111} + 3u_1^2) \cdot (u_3 - Q) - \frac{1}{2}(-u_2 + u_{111} + 3u_1^2) \cdot \frac{\mathrm{d}}{\mathrm{d}t_1}(u_3 - Q)$$

Double-zero property

If $\mathrm{d}\mathcal{L} = \sum_{i<j<k} (A^1_{ijk} B^1_{ijk} + \ldots + A^n_{ijk} B^n_{ijk})\, \mathrm{d}t^i \wedge \mathrm{d}t^j \wedge \mathrm{d}t^k$,

then the system $A^\ell_{ijk} = 0$, $B^\ell_{ijk} = 0$ implies the multiform Euler-Lagrange equations

Contents

From conservation law to multiform

So far:

$$\text{Lagrangian } L\text{, variational symmetry}$$
$$\Downarrow$$
$$\text{Lagrangian multiform } \mathcal{L}$$
$$\Downarrow$$
$$\text{Conservation law (Noether):} \quad \mathrm{d}\mathcal{L} = AB \qquad \text{where } A = \frac{\delta L}{\delta u}$$

What if we don't have a Lagrangian to start with?

Start from a conservation law for an equation $A = 0$, with characteristic B,

$$AB = \text{divergence},$$

without imposing that A is a variational derivative.

Mikhalev equation

$$u_{33} - u_{12} + u_3 u_{11} - u_1 u_{13} = 0$$

possesses four first-order and seven second-order conservation laws,

$$\mathcal{L} = \underbrace{L_{23}}_{F} \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \underbrace{L_{31}}_{G} \mathrm{d}x^3 \wedge \mathrm{d}x^1 + \underbrace{L_{12}}_{H} \mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

[VG Mikhalev. Functional Analysis and Its Applications, 1992]

[H Baran, IS Krasil'Shchik, OI Morozov, P Vojčák. Journal of Physics: Conference Series, 2014]

For any such conservation law, $\mathrm{d}\mathcal{L} = (F_1 + G_2 + H_3)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ factorises as

$$F_1 + G_2 + H_3 = (u_{33} - u_{12} + u_3 u_{11} - u_1 u_{13}) \cdot B[u],$$

where the characteristic B is a differential expression in u such that the system

$$u_{33} - u_{12} + u_3 u_{11} - u_1 u_{13} = 0, \qquad B[u] = 0$$

is involutive (i.e. compatible, i.e. multidimensionally consistent).

Thus, these conservation laws $\mathcal{L}$ are Lagrangian multiforms.

Mikhalev equation: Lagrangian 2-form for 3d PDE and ODE

Take, for example,

$$
\begin{aligned}
F &= u_3 u_{11}^2 - u_{13}^2 - 2u_{11}(u_{33} - u_{12} + u_3 u_{11} - u_1 u_{13}),\\
G &= -u_{11}^2,\\
H &= 2u_{11} u_{13} - u_1 u_{11}^2.
\end{aligned}
$$

Due to the factorisation

$$F_1 + G_2 + H_3 = -2(u_{33} - u_{12} + u_3 u_{11} - u_1 u_{13})\, u_{111}$$

the multiform Euler-Lagrange equations are consequences of the system

$$u_{33} - u_{12} + u_3 u_{11} - u_1 u_{13} = 0, \qquad u_{111} = 0.$$

The critical points are joint solutions of Mikhalev equation and the ODE $u_{111} = 0$.

Mikhalev equation: Lagrangian 2-form for 3d PDE and 2d PDE

This example can be deformed by adding to $\mathcal{L}$ a first-order conservation law of the Mikhalev equation:

$$\tilde{F} = u_3 u_{11}^2 - u_{13}^2 - 2u_{11}(u_{33} - u_{12} + u_3 u_{11} - u_1 u_{13}) + 2u_1 u_3^2 - u_1^3 u_3 - u_2 u_3,$$
$$\tilde{G} = -u_{11}^2 + u_1^3 - u_1 u_3,$$
$$\tilde{H} = 2u_{11} u_{13} - u_1 u_{11}^2 + u_1^4 - 3u_1^2 u_3 + u_1 u_2 + u_3^2,$$

with the factorisation

$$\tilde{F}_1 + \tilde{G}_2 + \tilde{H}_3 = 2\,(u_{33} - u_{12} + u_3 u_{11} - u_1 u_{13}) \left(u_3 - u_{111} - \frac{3}{2} u_1^2 \right).$$

The multiform Euler-Lagrange equations are equivalent to the system

$$u_{33} - u_{12} + u_3 u_{11} - u_1 u_{13} = 0, \qquad u_3 - u_{111} - \frac{3}{2} u_1^2 = 0,$$

so critical points are joint solutions of Mikhalev equation and potential KdV.

Mikhalev equation: Lagrangian 2-form for 3d PDE and 2d PDE

$$
\begin{aligned}
F &= 2u_1^3 u_{11} u_{13} - u_1^2 u_{11}^2 u_3 + 2u_1^2 u_{11} u_{12} - 3u_1^2 u_{13}^2 - 2u_1^2 u_{11} u_{33} - 2u_1 u_{12} u_{13} + u_{13}^2 u_3 \\
&\qquad + 4u_1 u_{13} u_{33} - u_{33}^2, \\
G &= -\left(u_1 u_{11} - u_{13}\right)^2, \\
H &= -u_1^3 u_{11}^2 + 2u_1^2 u_{11} u_{13} + 2u_1 u_{11}^2 u_3 - 2u_1 u_{11} u_{12} - u_1 u_{13}^2 - 2u_{11} u_{13} u_3 + 2\, u_{12} u_{13}.
\end{aligned}
$$

satisfies

$$F_1 + G_2 + H_3 = -2(u_{33} - u_{12} + u_3 u_{11} - u_1 u_{13})(u_1 u_{11}^2 + u_1^2 u_{111} - 2u_1 u_{113} - u_{11} u_{13} + u_{133}),$$

so $\mathcal{L}$ is a Lagrangian multiform for the involutive system

$$u_{33} - u_{12} + u_3 u_{11} - u_1 u_{13} = 0, \quad u_1 u_{11}^2 + u_1^2 u_{111} - 2u_1 u_{113} - u_{11} u_{13} + u_{133} = 0.$$

The second equation is equivalent to the Gurevich-Zybin equation for $v = u_1$,

$$(\partial_3 - v\,\partial_1)^2 v = 0.$$

Veronese Web hierarchy: Lagrangian 2-form for system of 3d PDEs

Consider
$$\mathcal{L} = \sum_{i<j} (c^i - c^j) \frac{u_{ij}^2}{u_i u_j} \, \mathrm{d}x^i \wedge \mathrm{d}x^j,$$

where $c^i = \text{const.}$ Degeneration of generating KdV multiform from [Lobb, Nijhoff 2009]

We have $\mathrm{d}\mathcal{L} = \sum_{i<j<k} A_{ijk} B_{ijk} \, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$ with

$$A_{ijk} = (c^i - c^j)\frac{u_{ij}}{u_i u_j} + (c^j - c^k)\frac{u_{jk}}{u_j u_k} + (c^k - c^i)\frac{u_{ik}}{u_i u_k},$$

$$B_{ijk} = u_{ijk} - \frac{1}{2}\left(\frac{u_{ij} u_{ik}}{u_i} + \frac{u_{ij} u_{jk}}{u_j} + \frac{u_{ik} u_{jk}}{u_{ik}} \right),$$

- Equations $A_{ijk} = 0$ form Veronese web hierarchy.

  see eg [M Dunajski, W Kryński, 2014]

- Equations $B_{ijk} = 0$ characterise potential (Egorov) metrics $\sum_i u_i (\mathrm{d}x^i)^2$ with diagonal curvature, i.e. all $R^i_{kkj} = 0$ for distinct i, j, k. see eg [VE Zakharov, 1998]

Veronese Web hierarchy: Lagrangian 2-form for system of 2d and 3d PDEs

The multiform Euler-Lagrange equations are

$$(1)\qquad \left(\frac{u_{ij}^2}{u_i^2 u_j}\right)_i + \left(\frac{u_{ij}^2}{u_i u_j^2}\right)_j + \left(\frac{2u_{ij}}{u_i u_j}\right)_{ij} = 0,$$

$$(2)\qquad (c^j - c^i)\left(\frac{u_{ij}^2}{u_i u_j^2} + 2\frac{u_{ii}u_{ij}}{u_i^2 u_j} - 2\frac{u_{iij}}{u_i u_j}\right) - (c^k - c^i)\left(\frac{u_{ik}^2}{u_i u_k^2} + 2\frac{u_{ii}u_{ik}}{u_i^2 u_k} - 2\frac{u_{iik}}{u_i u_k}\right) = 0,$$

$$(3)\qquad (c^i - c^j)\frac{u_{ij}}{u_i u_j} + (c^j - c^k)\frac{u_{jk}}{u_j u_k} + (c^k - c^i)\frac{u_{ik}}{u_i u_k} = 0.$$

This system is equivalent to the equations $A_{ijk} = 0$, $B_{ijk} = 0$, with

$$A_{ijk} = (c^i - c^j)\frac{u_{ij}}{u_i u_j} + (c^j - c^k)\frac{u_{jk}}{u_j u_k} + (c^k - c^i)\frac{u_{ik}}{u_i u_k},$$

$$B_{ijk} = u_{ijk} - \frac{1}{2}\left(\frac{u_{ij}u_{ik}}{u_i} + \frac{u_{ij}u_{jk}}{u_j} + \frac{u_{ik}u_{jk}}{u_i k}\right),$$

Recipe

Take an integrable PDE

$$A = 0$$

Consider a conservation law F, G, H such that

$$F_1 + G_2 + H_3 = A \cdot B.$$

Then the combined system

$$A = 0, \qquad B = 0$$

seems to be

- in involution (i.e. consistent/commuting)
- variational – it characterises stationary points of the Lagrangian multiform principle.

Under what conditions does this hold?

Contents

Conclusions (or lack thereof)

Also in the pre-print (arXiv:2503.22615)

- Gibbons-Tsarev type equations for hydrodynamic reductions of second heavenly equation. (Similar ideas, several dependent variables).
- Non-translation-invariant version of VWE.

Key ideas

- Lagrangian 2-forms can describe 3d PDEs.
- The closure property of Lagrangian multiforms naturally relates to conservation laws.
- This seems to be the case even if a priori there is no Lagrangian structure.

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Thank you for your attention!