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Contact variational integrators

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My introduction to contact mechanics (2018)

Main references:

[V, Bravetti, Seri. Contact variational integrators. J Phys A, 2019]

[Anahory Simoes, Martín de Diego, Lainz Valcázar, de León. On the Geometry of Discrete Contact Mechanics. JNLS, 2021]

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Background: symplectic dynamics

Symplectic form ω: closed non-degenerate 2-form on a $2n$-dimensional manifold $\mathcal{M}$

A Hamilton function $H : \mathcal{M} \to \mathbb{R}$ induces a Hamiltonian vector field X_H on $\mathcal{M}$:

$$\iota_{X_H}\omega = \mathrm{d}H$$

In Darboux coordinates $(x_1, \dots, x_n, p_1, \dots p_n)$

$$\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}x_i$$

The vector field X_H is given by

$$\dot{x} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial x}.$$

Example. In mechanics, we usually have $H(x,p) = \frac{1}{2}|p|^2 + U(x)$ leading to

$$\dot{x} = p, \qquad \dot{p} = -U'(x)$$

Properties of Hamiltonian systems

The flow of $F_t : \mathcal{M} \to \mathcal{M} : (x(0), p(0)) \mapsto (x(t), p(t))$ of a Hamiltonian vector field preserves the symplectic form,

$$(F_t)^* \omega = \omega,$$

the corresponding volume

$$(F_t)^* \omega^n = \omega^n \qquad = \mathrm{d}p_1 \wedge \ldots \wedge \mathrm{d}p_n \wedge x_1 \wedge \ldots \wedge \mathrm{d}x_n,$$

and the energy,

$$H(x(t), p(t)) = H(x(0), p(0)).$$

If the system has symmetries, then the corresponding generalized momenta are conserved quantities (Noether's theorem).

Lagrangian mechanics

If we can solve $\dot{x} = \frac{\partial H}{\partial p}$ for p, then solutions to the Hamiltonian equations satisfy a variational principle:

$$\delta \int_0^t \mathcal{L}(x, \dot{x}) \, \mathrm{d}t = 0$$

for variations δx of x leaving the endpoints $x(0)$ and $x(t)$ invariant, where the Lagrangian is $\mathcal{L}(x, \dot{x}) = p\dot{x} - H(x, p)$.

Critical curves are characterized by Euler-Lagrange equation

$$0 = \int_0^t \frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}} \delta \dot{x} \, \mathrm{d}t = \int_0^t \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x \, \mathrm{d}t \qquad \Leftrightarrow \qquad \frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

Example. For $H(x, p) = \frac{1}{2}|p|^2 + U(x)$ we find

$$\mathcal{L}(x, \dot{x}) = |\dot{x}|^2 - U(x)$$

leading to the Euler-Lagrange equation $-U'(x) - \ddot{x} = 0$

Geometric discretisation

Main idea

Discretisation preserving the geometric structure often leads to improved accuracy, especially over long time intervals.

A map $\Phi_h : \mathcal{M} \to \mathcal{M}, \Phi_h(x,p) = (x,p) + \mathcal{O}(h)$ is a consistent discretisation of the flow F_t if

$$\Phi_h(x,p) = F_h(x,p) + \mathcal{O}(h^2) \qquad (= (x,p) + \mathcal{O}(h))$$

Φ_h is called symplectic if it preserves ω: $\quad (\Phi_h)^*\omega = \omega.$

An effective way to obtain sympletic integrators is by discretizing the variational principle:

Look for a discrete curve $x_0, x_1, \ldots, x_N$ minimizing the discrete action

$$\textstyle\sum_i L(x_i, x_{i+1}; h),$$

where $L(x(0), x(h); h) \approx \min_x \int_0^h \mathcal{L}(x(t), \dot{x}(t))\,\mathrm{d}t.$

Properties of symplectic integrators

By definition, a symplectic integrator preserves the symplectic form,

$$(\Phi_h)^*\omega = \omega,$$

and hence the corresponding volume

$$(\Phi_h)^*\omega^n = \omega^n.$$

A symplectic integrator very nearly preserves a modified energy $E_{mod} \approx H$:

$$E_{mod}(\Phi_h^n(x,p)) \approx E_{mod}(x,p)$$

over a time interval of length $\mathcal{O}(e^{-h})$.

If the discretization has symmetries, then there exist conserved generalized discrete momenta.

[Marsden, West. Discrete mechanics and variational integrators. Acta numerica, 2001]

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Contact geometry on a $(2n+1)$-dimensional manifold M

A Contact structure is a distribution of hyperplanes $\xi \subset TM$ that is maximally non-integrable: a submanifold tangent to the distribution has dimension at most n.

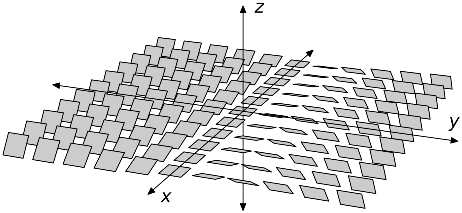
Locally, such a distribution is given by the kernel of a 1-form η on M satisfying

$$\eta \wedge (\mathrm{d}\eta)^n \neq 0.$$

Reeb vector field R defined by

$$\iota_R \mathrm{d}\eta = 0 \quad \text{and} \quad \eta(R) = 1.$$

Multiplying η by a non-vanishing function does not change the contact structure.



$\mathrm{d}z - y\mathrm{d}x$

$F : M \to M$ is a contact transformation if $F^*\eta = g\eta$ for some $g : M \to \mathbb{R} \setminus \{0\}$.

$X : M \to TM$ is a contact vector field if $\mathscr{L}_X \eta = g\eta$ for some $g : M \to \mathbb{R} \setminus \{0\}$, where $\mathscr{L}$ denotes the Lie derivative

Contact Hamiltonian systems

There exist local coordinates $(x_1, \dots, x_n, p_1, \dots, p_n, z)$ such that the contact 1-form is

$$\eta = \mathrm{d}z - p\,\mathrm{d}x \qquad = \mathrm{d}z - \textstyle\sum_i p_i\,\mathrm{d}x_i,$$

and the Reeb vector field is $R = \partial_z$.

Contact Hamiltonian vector field

$$\mathscr{L}_{X_H}\eta = g_H\,\eta \qquad \text{and} \qquad \eta(X_H) = -H,$$

where $g_H : M \to \mathbb{R} \setminus \{0\}$ is a scalar function, given by $g_H = -R_\eta(H)$.

For comparison with symplectic mechanics, note that

$$\iota_{X_H}(\mathrm{d}p \wedge \mathrm{d}q) = \iota_{X_H}(\mathrm{d}\eta) = -\mathrm{d}(\iota_{X_H}\eta) + \mathcal{L}_X\eta = \mathrm{d}H + g_H\,\eta.$$

In Darboux coordinates the contact Hamiltonian equations are

$$\dot{x} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial x} - p\frac{\partial H}{\partial z}, \qquad \dot{z} = p\frac{\partial H}{\partial p} - H.$$

Damped mechanical systems

Contact Hamiltonian systems satisfy

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -H\frac{\partial H}{\partial z} \qquad \leftrightarrow \qquad X_H H = -R(H)H$$

so dissipation can occur!

Example. A Hamiltonian of the form

$$H = \frac{1}{2}p^2 + U(x) + \alpha z$$

describes a mechanical system with linear damping:

$$\begin{cases} \dot{x} = p \\ \dot{p} = -U'(x) - \alpha p \\ \dot{z} = p^2 - H. \end{cases}$$

Written as a second order ODE: $\ddot{x} = -U'(x) - \alpha\dot{x}$.

Other applications

- Thermodynamics

  [Bravetti. Contact geometry and thermodynamics. International Journal of Geometric Methods in Modern Physics, 2018.]

- Integrable systems

  [Sergyeyev. New integrable $(3 + 1)$-dimensional systems and contact geometry. Letters in Mathematical Physics, 2018.]

- Optimal control

  [Ohsawa T. Contact geometry of the Pontryagin maximum principle. Automatica, 2015.]

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Hamiltonian integrators

In many examples, $H(x,p,z) = A(p) + B(x) + Cz$. Then

$$\begin{aligned} X_A &= A'(p)\partial_x + (pA(p) - A(p))\partial_z \\ X_B &= -B'(x)\partial_p - B(x)\partial_z \\ X_{Cz} &= -pC\partial_p - Cz\partial_z, \end{aligned}$$

which are all explicitly integrable:

$$\begin{aligned} \exp(tX_A)(x,p,z) &= (x + tA'(p), p, z + t(pA(p) - A(p)) \\ \exp(tX_B)(x,p,z) &= (x, p - t(B'(x) + B(x)), z + t(pA(p) - A(p)) \\ \exp(tX_C)(x,p,z) &= (x, p - tpC, \exp(Ct)z) \end{aligned}$$

Splitting integrator (2nd order)

$$S_2(h) = \exp\left(\frac{h}{2}X_C\right)\exp\left(\frac{h}{2}X_B\right)\exp(hX_A)\exp\left(\frac{h}{2}X_B\right)\exp\left(\frac{h}{2}X_C\right).$$

As a composition of contact transformations, $S_2(h)$ is itself a contact transformation.

Hamiltonian integrators

Given a second order contact integrator S_2, higher order contact integrators can be obtained recursively by "Yoshida's trick":

$$S_{2n+2}(h) = S_{2n}(\alpha_n h) S_{2n}(\beta_n h) S_{2n}(\alpha_n h)$$

where $\alpha_n = \frac{1}{2-2^{\frac{1}{2n+1}}}$ and $\beta_n = -\frac{2^{\frac{1}{2n+1}}}{2-2^{\frac{1}{2n+1}}}$.

A more complicated but similar construction for S_2 applies for Hamiltonians

$$H(t,x,p,z) = A(t,p) + B(t,x) + C(t)z$$

depending explicitly on time.

[Yoshida. Construction of higher order symplectic integrators. Physics letters A, 1990]

[Bravetti, Seri, V, Zadra. Numerical integration in celestial mechanics: a case for contact geometry. Celest Mech Dyn Astr, 2020]

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Herglotz' variational principle

The contact Hamiltonian equation for z is

$$\dot{z} = p\frac{\partial H}{\partial p} - H \qquad \stackrel{?}{=} \mathcal{L}$$

Herglotz' variational principle

Lagrangian $\mathcal{L} : TQ \times \mathbb{R} \to \mathbb{R}$.
Given a curve $x : [0, T] \to Q$, define $z : [0, T] \to \mathbb{R}$ by $z(0) = z_0$ and

$$\dot{z}(t) = \mathcal{L}(x(t), \dot{x}(t), z(t))$$

We look for a curve x such that every variation of x that vanishes at the boundary of $[0, T]$ leaves the action $z(T)$ invariant.

If $\mathcal{L}$ does not depend on z we find the familiar action: $z(T) = \int_0^T \mathcal{L}(x(t), \dot{x}(t))\,\mathrm{d}t$.

[Herglotz. Berührungstransformationen Lecture notes, Göttingen, 1930.]

Direct approach aka implicit approach

A variation δx of x induces a variation δz of z:

$$\dot z(t) = \mathcal{L}(x(t), \dot x(t), z(t)) \quad \Rightarrow \quad \delta \dot z = \underbrace{\frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \dot x}\delta \dot x}_{A(t)} + \underbrace{\frac{\partial \mathcal{L}}{\partial z}}_{\frac{\mathrm{d}B(t)}{\mathrm{d}t}}\delta z.$$

The solution of $\delta \dot z(t) = A(t) + \dfrac{\mathrm{d}B(t)}{\mathrm{d}t}\delta z(t)$ is

$$\begin{aligned}
\delta z(T) &= e^{B(T)}\left[\int_0^T A(\tau) e^{-B(\tau)}\,\mathrm{d}\tau + \delta z(0)\right] \\
&= e^{B(T)}\left[\int_0^T \left(\frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \dot x}\delta \dot x\right) e^{-B(\tau)}\,\mathrm{d}\tau + \delta z(0)\right] \\
&= e^{B(T)}\Bigg[\int_0^T \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot x} + \frac{\partial \mathcal{L}}{\partial z}\frac{\partial \mathcal{L}}{\partial \dot x}\right)\delta x\, e^{-B(\tau)}\,\mathrm{d}\tau \\
&\qquad + \frac{\partial \mathcal{L}}{\partial \dot x}(T)\delta x(T)\, e^{-B(T)} - \frac{\partial \mathcal{L}}{\partial \dot x}(0)\delta x(0) + \delta z(0)\Bigg].
\end{aligned}$$

Direct approach aka implicit approach

$$\delta z(T) = e^{B(T)} \left[\int_0^T \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} + \frac{\partial \mathcal{L}}{\partial z} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x \, e^{-B(\tau)} \, \mathrm{d}\tau \right.$$
$$\left. + \frac{\partial \mathcal{L}}{\partial \dot{x}}(T) \delta x(T) \, e^{-B(T)} - \frac{\partial \mathcal{L}}{\partial \dot{x}}(0) \delta x(0) + \delta z(0) \right].$$

Variations satisfy $\delta x(0) = \delta x(T) = \delta z(0) = 0$.

Generalized Euler-Lagrange equations: $$\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} + \frac{\partial \mathcal{L}}{\partial z} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

If instead we restrict to solution curves, but vary the endpoints, we obtain

$$\delta z(T) = \frac{\partial \mathcal{L}}{\partial \dot{x}}(T) \delta x(T) - e^{B(T)} \left[\frac{\partial \mathcal{L}}{\partial \dot{x}}(0) \delta x(0) + \delta z(0) \right]$$

Contact structure: $\phi_T^*(\mathrm{d}z - p \, \mathrm{d}x) = e^{B(T)} (\mathrm{d}z - p \, \mathrm{d}x)$
where $p = \frac{\partial \mathcal{L}}{\partial \dot{x}}$ and ϕ_T denotes the flow over the time interval $[0, T]$.

Vakonomic approach

Consider $\dot z = \mathcal{L}$ as a constraint, so the action becomes

$$S = z(T) + \int_0^T \lambda\big(\dot z - \mathcal{L}(x,\dot x, z)\big)\,\mathrm{d}t \qquad = z(0) + \int_0^T \dot z + \lambda\big(\dot z - \mathcal{L}(x,\dot x, z)\big)\,\mathrm{d}t$$

Vary x, z with fixed endpoints:

$$0 = \frac{\delta S}{\delta x} = \lambda \frac{\partial \mathcal{L}}{\partial x} - \lambda \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot x} - \dot\lambda \frac{\partial \mathcal{L}}{\partial \dot x}$$
$$0 = \frac{\delta S}{\delta z} = -\dot\lambda - \lambda\frac{\partial \mathcal{L}}{\partial z}$$

Hence $\quad \dot\lambda = -\lambda \frac{\partial \mathcal{L}}{\partial z} \quad$ and $\quad \lambda\frac{\partial \mathcal{L}}{\partial x} - \lambda\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot x} + \lambda \frac{\partial \mathcal{L}}{\partial z}\frac{\partial \mathcal{L}}{\partial \dot x} = 0$

If we restrict to solution curves, but vary the endpoints, we get $\delta S = \Big[\lambda(\delta z - p\delta x)\Big]_0^T$.

[León, Lainz, Muñoz-Lecanda. The Herglotz principle and vakonomic dynamics. International Conference on Geometric Science of Information, 2021]

Dissipated quantities

If $v = (v_x \partial_x + v_z \partial_z)$ is a variational symmetry, i.e.

$$\delta S(v) = 0 \qquad \text{for all curves } x, z, \text{ not just solutions,}$$

then restricted to solutions we find

$$\delta S(v) = \Big[\lambda(\delta z - p\delta x)\Big]_0^T (v) = \Big[\lambda(v_z - p v_x)\Big]_0^T = 0$$

so $\lambda \eta(v) = \lambda(v_z - p v_x)$ is constant.

$$\eta(v) = v_z - p v_x \sim \lambda^{-1} \qquad \text{is a dissipated quantity.}$$

λ does not depend on v, so each dissipated quantity has the same rate of dissipation.

In particular, if $v = \partial_t$, i.e. $v_x = \dot{x}$ and $v_z = \dot{z}$, we find the dissipated quantity

$$\dot{z} - p\dot{x} = \mathcal{L} - p\dot{x} = -H.$$

Equivalently, if $v = X_H$, then the dissipated quantity is $\eta(X_H) = -H$.

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Discrete Herglotz variational principle

Variational integrator: approximate $\min_x \int_t^{t+h} \mathcal{L}(x, \dot{x}, z)\, \mathrm{d}t$ by $L(x_j, x_{j+1}, z_j, z_{j+1}; h)$, where $h > 0$ is the step size.

Discrete Herglotz variational principle

Given $x = x_0, x_1, x_2 \ldots \in Q$ we define $z = z_0, z_1, z_2 \in \mathbb{R}$ by $z_0 = 0$ and

$$z_{j+1} - z_j = hL(x_j, x_{j+1}, z_j, z_{j+1}; h) \qquad (*)$$

Fix a final "time" N and the values of x_0 and x_N. Then look for a discrete curve x such that

$$\frac{\mathrm{d}z_N}{\mathrm{d}x_j} = 0 \qquad \forall j \in \{1, \ldots, N-1\}$$

Solving $(*)$ for z_{j+1} we obtain a Lagrangian $\tilde{L}(x_j, x_{j+1}, z_j; h)$ that does not depend on z_{j+1}:

$$z_{j+1} = z_j + h\tilde{L}(x_j, x_{j+1}, z_j; h) \qquad (\dagger)$$

Direct approach aka implicit approach

From

$$z_j = z_{j-1} + hL(x_{j-1}, x_j, z_{j-1}) \qquad (\dagger)$$

it follows that variations wrt x_i evolve as

$$\frac{\partial z_j}{\partial x_i} = \frac{\partial z_{j-1}}{\partial x_i} + h\mathrm{D}_1 L(x_{j-1}, x_j, z_{j-1})\frac{\partial x_{j-1}}{\partial x_i} + h\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1})\frac{\partial x_j}{\partial x_i} + h\mathrm{D}_3 L(x_{j-1}, x_j, z_{j-1})\frac{\partial z_{j-1}}{\partial x_i}$$

$$= \left(1 + h\mathrm{D}_3 L(x_{j-1}, x_j, z_{j-1})\right)\frac{\partial z_{j-1}}{\partial x_i} + h\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1})\delta_j^i + h\mathrm{D}_1 L(x_{j-1}, x_j, z_{j-1})\delta_{j-1}^i,$$

where D_i denotes partial derivative wrt i-th entry.

This implies that

Lemma

For h sufficiently small

$$\frac{\partial z_N}{\partial x_i} = 0 \iff \frac{\partial z_{i+1}}{\partial x_i} = 0$$

Direct approach aka implicit approach

From $z_i = z_{i-1} + hL(x_{i-1}, x_i, z_{i-1})$ we first compute $\dfrac{\partial z_i}{\partial x_i} = h\mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1})$

Then from $z_{i+1} = z_i + hL(x_i, x_{i+1}, z_i)$ we obtain

$$\begin{aligned}\frac{\partial z_{i+1}}{\partial x_i} &= \frac{\partial z_i}{\partial x_i} + h\mathrm{D}_1 L(x_i, x_{i+1}, z_i) + h\mathrm{D}_3 L(x_i, x_{i+1}, z_i)\frac{\partial z_i}{\partial x_i} \\ &= h\mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) + h\mathrm{D}_1 L(x_i, x_{i+1}, z_i) + h^2\mathrm{D}_3 L(x_i, x_{i+1}, z_i)\mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1})\end{aligned}$$

Discrete generalized Euler-Lagrange equation

$$0 = \mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) + \mathrm{D}_1 L(x_i, x_{i+1}, z_i) + h\mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1})\mathrm{D}_3 L(x_i, x_{i+1}, z_i).$$

If L depends on z_j and z_{j+1}

$$\begin{aligned}0 = {} & \mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}, z_j) + \mathrm{D}_1 L(x_j, x_{j+1}, z_j, z_{j+1}) \\ & + \tfrac{h\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}, z_j)}{1 - h\mathrm{D}_4 L(x_{j-1}, x_j, z_{j-1}, z_j)}(\mathrm{D}_3 L(x_j, x_{j+1}, z_j, z_{j+1}) + \mathrm{D}_4 L(x_{j-1}, x_j, z_{j-1}, z_j)).\end{aligned}$$

Discrete Herglotz variational principle

Discrete generalized Euler-Lagrange equation

$$
\begin{aligned}
0 &= \mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}, z_j) + \mathrm{D}_1 L(x_j, x_{j+1}, z_j, z_{j+1}) \\
&\quad + \tfrac{h \mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}, z_j)}{1 - h \mathrm{D}_4 L(x_{j-1}, x_j, z_{j-1}, z_j)} \big(\mathrm{D}_3 L(x_j, x_{j+1}, z_j, z_{j+1}) + \mathrm{D}_4 L(x_{j-1}, x_j, z_{j-1}, z_j) \big).
\end{aligned}
$$

where D_i is the partial derivative w.r.t. the i-th variable.

If L a consistent discretization of a continuous Lagrangian $\mathcal{L}$,

$$
\begin{aligned}
\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}, z_j) + \mathrm{D}_1 L(x_j, x_{j+1}, z_j, z_{j+1}) &\approx \frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} \\
\frac{h \mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}, z_j)}{1 - h \mathrm{D}_4 L(x_{j-1}, x_j, z_{j-1}, z_j)} &\approx \frac{\partial \mathcal{L}}{\partial \dot{x}} \\
\mathrm{D}_3 L(x_j, x_{j+1}, z_j, z_{j+1}) + \mathrm{D}_4 L(x_{j-1}, x_j, z_{j-1}, z_j) &\approx \frac{\partial \mathcal{L}}{\partial z}
\end{aligned}
$$

Contact structure

The discrete generalized Euler-Lagrange equation can be written as

$$\frac{h\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}, z_j)}{1 - h\mathrm{D}_4 L(x_{j-1}, x_j, z_{j-1}, z_j)} + \frac{h\mathrm{D}_1 L(x_j, x_{j+1}, z_j, z_{j+1})}{1 + h\mathrm{D}_3 L(x_j, x_{j+1}, z_j, z_{j+1})} = 0$$

Position-momentum formulation

$$\Phi : T^*Q \times \mathbb{R} \mapsto T^*Q \times \mathbb{R} : (x_{j-1}, p_{j-1}, z_{j-1}) \mapsto (x_j, p_j, z_j),$$

where $p_j = p_j^- = p_j^+$ and

$$p_j^- = \frac{h\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}, z_j)}{1 - h\mathrm{D}_4 L(x_{j-1}, x_j, z_{j-1}, z_j)},$$

$$p_j^+ = -\frac{h\mathrm{D}_1 L(x_j, x_{j+1}, z_j, z_{j+1})}{1 + h\mathrm{D}_3 L(x_j, x_{j+1}, z_j, z_{j+1})}.$$

The map Φ is a contact transformation with respect to the 1-form

Vakomic approach

$$S = z_0 + \sum_j (z_{j+1} - z_j) + \lambda_j \big(z_{j+1} - z_j - hL(x_j, x_{j+1}, z_j) \big)$$

Vary both x and z:

$$0 = \frac{\partial S}{\partial x_i} = -\lambda_i h \mathrm{D}_1 L(x_i, x_{i+1}, z_i) - \lambda_{i-1} h \mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1})$$

$$0 = \frac{\partial S}{\partial z_i} = \lambda_{i-1} - \lambda_i - \lambda_i h \mathrm{D}_3 L(x_i, x_{i+1}, z_i)$$

So

$$-\lambda_i h \mathrm{D}_1 L(x_i, x_{i+1}, z_i) - (\lambda_i + \lambda_i h \mathrm{D}_3 L(x_i, x_{i+1}, z_i))\, h \mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) = 0$$

and hence

$$\mathrm{D}_1 L(x_i, x_{i+1}, z_i) + (1 + h \mathrm{D}_3 L(x_i, x_{i+1}, z_i))\, \mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) = 0$$

$$0 = \mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) + \mathrm{D}_1 L(x_i, x_{i+1}, z_i) + h \mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) \mathrm{D}_3 L(x_i, x_{i+1}, z_i).$$

All contact maps are variational

Theorem

Iterations of any contact transformation

$$(x_0, p_0, z_0) \mapsto (x_1, p_1, z_1)$$

yield a discrete curve $x = (x_0, \ldots, x_N)$ that solves the discrete Herglotz variational principle for some discrete Lagrangian $L(x_j, x_{j+1}, z_j)$.

Proof idea. Like in the symplectic case, every contact transformation has a generating function, which can be used as a discrete Lagrangian. ∎

Backward error analysis

Solutions of the difference equations

$$\begin{cases} \dfrac{z_{j+1} - z_j}{h} = L(x_j, x_{j+1}, z_j, z_{j+1}; h) \\ \dfrac{x_{j+1} - 2x_j + x_{j-1}}{h^2} = F(x_{j-1}, x_j, x_{j+1}, z_{j-1}, z_j, z_{j+1}; h). \end{cases}$$

are formally interpolated by solutions of the modified equations

$$\begin{cases} \dot{z} = \mathcal{L}_{\text{mod}}(x, \dot{x}, z, h) = \mathcal{L}(x, \dot{x}, z) + h\mathcal{L}_1(x, \dot{x}, z) + h^2\mathcal{L}_2(x, \dot{x}, z) + \dots \\ \ddot{x} = f_{\text{mod}}(x, \dot{x}, z; h) = f(x, \dot{x}, z) + hf_1(x, \dot{x}, z) + h^2 f_2(x, \dot{x}, z) + \dots \end{cases}.$$

The modified equations are also a contact system

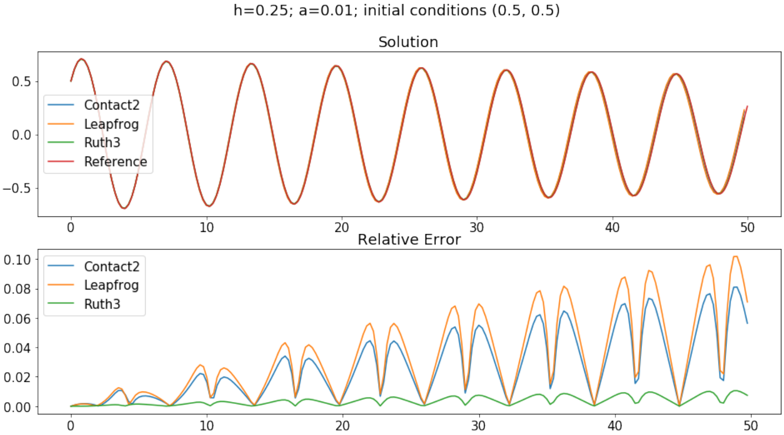
In particular, $\ddot{x} = f_{\text{mod}}(x, \dot{x}, z; h)$ is the generalized Euler-Lagrange equation of $\mathcal{L}_{\text{mod}}(x, \dot{x}, z, h)$.

The power series are usually not convergent. Truncations need to be used to make rigorous statements about long-time error bounds...

Numerical example: harmonic oscillator

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 - \alpha z \qquad \Rightarrow \qquad \ddot{x} = -x - \alpha \dot{x}$$

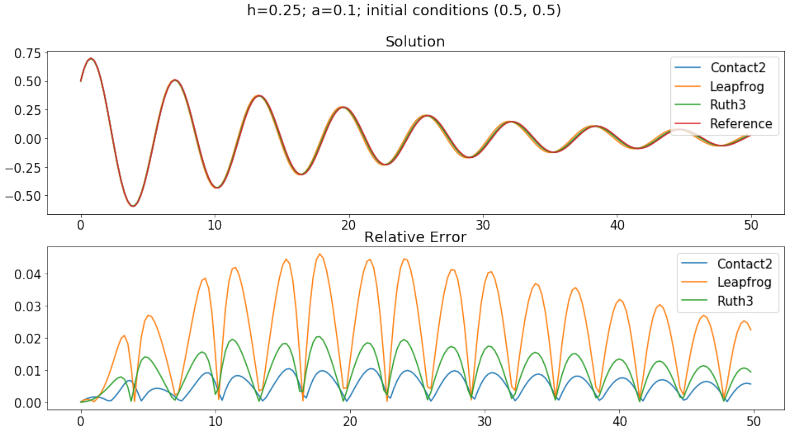
Very small damping: contact integrators comparable to symplectic integrators



Numerical example: harmonic oscillator

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 - \alpha z \qquad \Rightarrow \qquad \ddot{x} = -x - \alpha\dot{x}$$

Slightly larger damping: contact integrators better than symplectic integrators

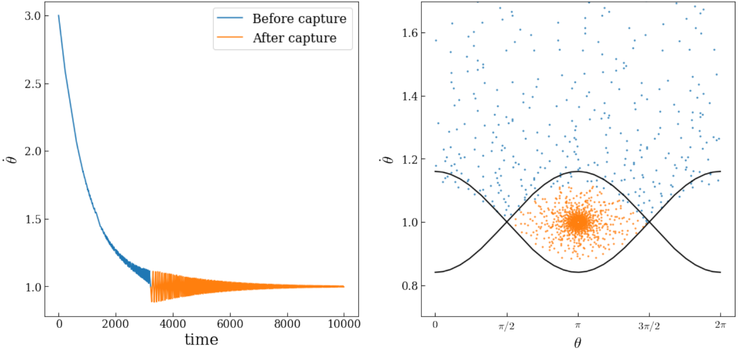


Time-depenent example: spin-orbit mechanics

Flexible satellite in a fixed orbit, experiencing torque from gravity.

The torque is a time-dependent linear dissipation:

$$H = \frac{p^2}{2} + \frac{N_z(\theta, t)}{C} + \frac{dC}{dt}\frac{1}{C} z \quad \Rightarrow \quad \ddot{\theta} + \frac{dC}{dt}\frac{\dot{\theta}}{C} - \frac{N_z(\theta, t)}{C} = 0.$$



Conclusions

- Contact mechanics is less known than symplectic mechanics, but has significant applications in physics and a similarly rich structure.
- Contact mechanics is described by Herglotz' principle, an variational principle with and "action-dependent" Lagrangian.
- Structure-preserving discretizations for contact systems can be obtained using many of the same ideas as for symplectic systems.
- Satisfying theory, but relevance in numerical integration likely more limited than symplectic integrators.

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