The discrete Herglotz variational principle

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Contact Hamiltonian mechanics

Contact geometry: odd-dimensional analogue to symplectic geometry.

Contact Hamiltonian mechanics: allows dissipation of energy

Variational principle

[Herglotz. Beruhrungstransformationen. 1930.]

[Georgieva, Guenther, Bodurov. Generalized variational principle of Herglotz for several independent variables. First Noether-type theorem. 2003]

[León, Lainz, Muñoz-Lecanda. The Herglotz principle and vakonomic dynamics. 2021]

[Gaset, Lainz, Mas, Rivas. The Herglotz variational principle for dissipative field theories. 2024]

Discretisation

[V, Bravetti, Seri. Contact variational integrators. 2019.]

[Bravetti, Seri, V, Zadra. Numerical integration in celestial mechanics: a case for contact geometry. 2020.]

[Bravetti, Seri, Zadra. New directions for contact integrators. 2021.]

[Anahory Simoes, Martín de Diego, Lainz Valcázar, de León. On the Geometry of Discrete Contact Mechanics. 2021]

Contact geometry

$(2n+1)$-dimensional manifold M.

Contact structure

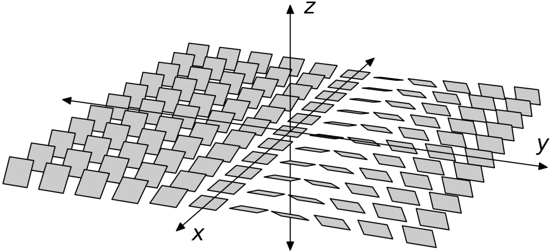
A distribution of hyperplanes $\xi \subset TM$ that is maximally non-integrable:
a submanifold that is always tangent to the distribution has dimension at most n.

Locally, such a distribution is given by the kernel of a 1-form η satisfying

$$\eta \wedge (\mathrm{d}\eta)^n \neq 0,$$

called a contact form.

Multiplying η by a non-vanishing function does not change the contact structure.



$\mathrm{d}z - y\mathrm{d}x$

$f : M \to M$ is a contact transformation if $f^*\eta = g\eta$ for some $g : M \to \mathbb{R}$.

Contact geometry

There exist Darboux local coordinates $(x_1, \dots, x_n, p_1, \dots, p_n, z)$ such that the contact 1-form can be written as

$$\eta = \mathrm{d}z - p\,\mathrm{d}x \qquad = \mathrm{d}z - \sum_i p_i\,\mathrm{d}x_i.$$

Contact Hamiltonian vector field

$$\mathcal{L}_{X_H}\eta = f_H\eta \qquad \text{and} \qquad \eta(X_H) = -H,$$

where $\mathcal{L}$ is Lie derivative and $f_H : M \to \mathbb{R}$. $(f_H = -R_\eta(H)$, where R_η is the Reeb vector field.)

For comparison with symplectic mechanics, note that

$$\iota_{X_H}(\mathrm{d}\eta) = -\mathrm{d}(\iota_{X_H}\eta) + \mathcal{L}_X\eta = \mathrm{d}H + f_H\eta.$$

In Darboux coordinates the contact Hamiltonian equations are

$$\dot{x} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial x} - p\frac{\partial H}{\partial z}, \qquad \dot{z} = p\frac{\partial H}{\partial p} - H.$$

Damped mechanical systems

Contact Hamiltonian systems satisfy $$\frac{\mathrm{d}H}{\mathrm{d}t} = -H\frac{\partial H}{\partial z}$$ so dissipation can occur!

Example. A Hamiltonian of the form

$$H = \frac{1}{2}p^2 + V(x) + \alpha z$$

describes a mechanical system with linear damping:

$$\begin{cases} \dot{x} = p \\ \dot{p} = -V'(x) - \alpha p \\ \dot{z} = p^2 - H. \end{cases}$$

Written as a second order ODE:

$$\ddot{x} = -V'(x) - \alpha \dot{x}.$$

(Meaning of z will be discussed later.)

Other applications

Thermodynamics [Bravetti. Contact geometry and thermodynamics. International Journal of Geometric Methods in Modern Physics, 2018.]

Integrable systems [Sergyeyev. New integrable $(3 + 1)$-dimensional systems and contact geometry. Letters in Mathematical Physics, 2018.]

Optimal control [Ohsawa T. Contact geometry of the Pontryagin maximum principle. Automatica, 2015.]

Liénard Systems (e.g. van der Pol oscillator) [Zadra, Bravetti, Seri. Geometric Numerical Integration of Liénard Systems via a Contact Hamiltonian Approach. MDPI Mathematics, 2021.]

Scaling reductions [Bravetti, Jackman, Sloan. Scaling symmetries, contact reduction and Poincaré's dream. J Phys A, 2023.]

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Herglotz' variational principle

The contact Hamiltonian equation for z is

$$\dot{z} = p\frac{\partial H}{\partial p} - H \qquad \stackrel{?}{=} \mathcal{L}$$

Herglotz' variational principle [Herglotz, 1930]

Lagrangian $\mathcal{L} : TQ \times \mathbb{R} \to \mathbb{R}$.
Given a curve $x : [0, T] \to Q$, define $z : [0, T] \to \mathbb{R}$ by $z(0) = z_0$ and

$$\dot{z}(t) = \mathcal{L}(x(t), \dot{x}(t), z(t))$$

We look for a curve x such that every variation of x that vanishes at the boundary of $[0, T]$ leaves the action $z(T)$ invariant.

If $\mathcal{L}$ does not depend on z we find the classical variational principle:

$$z(T) = \int_0^T \mathcal{L}(x(t), \dot{x}(t))\,\mathrm{d}t.$$

Direct approach (aka implicit approach)

A variation δx of x induces a variation δz of z:

$$\dot z(t) = \mathcal{L}(x(t), \dot x(t), z(t)) \quad \Rightarrow \quad \delta \dot z = \underbrace{\frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \dot x}\delta \dot x}_{A(t)} + \underbrace{\frac{\partial \mathcal{L}}{\partial z}}_{\frac{\mathrm{d}B(t)}{\mathrm{d}t}} \delta z.$$

The solution of $\delta \dot z(t) = A(t) + \frac{\mathrm{d}B(t)}{\mathrm{d}t}\delta z(t)$ is

$$\begin{aligned}
\delta z(T) &= e^{B(T)} \left[\int_0^T A(\tau) e^{-B(\tau)} \,\mathrm{d}\tau + \delta z(0) \right] \\
&= e^{B(T)} \left[\int_0^T \left(\frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \dot x}\delta \dot x \right) e^{-B(\tau)} \,\mathrm{d}\tau + \delta z(0) \right] \\
&= e^{B(T)} \Bigg[\int_0^T \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot x} + \frac{\partial \mathcal{L}}{\partial z}\frac{\partial \mathcal{L}}{\partial \dot x} \right) \delta x \, e^{-B(\tau)} \,\mathrm{d}\tau \\
&\qquad + \frac{\partial \mathcal{L}}{\partial \dot x}(T)\delta x(T)\, e^{-B(T)} - \frac{\partial \mathcal{L}}{\partial \dot x}(0)\delta x(0) + \delta z(0) \Bigg].
\end{aligned}$$

Direct approach (aka implicit approach)

$$\delta z(T) = e^{B(T)}\left[\int_0^T \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} + \frac{\partial \mathcal{L}}{\partial z}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\delta x\, e^{-B(\tau)}\,\mathrm{d}\tau \right.$$
$$\left. + \frac{\partial \mathcal{L}}{\partial \dot{x}}(T)\delta x(T)\, e^{-B(T)} - \frac{\partial \mathcal{L}}{\partial \dot{x}}(0)\delta x(0) + \delta z(0)\right].$$

Variations satisfy $\delta x(0) = \delta x(T) = \delta z(0) = 0$.

Generalized Euler-Lagrange equations
$$\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} + \frac{\partial \mathcal{L}}{\partial z}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

If instead we restrict to solution curves, but vary the endpoints, we obtain

$$\delta z(T) = \frac{\partial \mathcal{L}}{\partial \dot{x}}(T)\delta x(T) - e^{B(T)}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}}(0)\delta x(0) + \delta z(0)\right]$$

Contact structure
$$\phi_T^*(\mathrm{d}z - p\,\mathrm{d}x) = e^{B(T)}(\mathrm{d}z - p\,\mathrm{d}x)$$

where $p = \frac{\partial \mathcal{L}}{\partial \dot{x}}$ and ϕ_T denotes the flow over the time interval $[0, T]$.

Vakonomic approach

Consider $\dot z = \mathcal{L}$ as a constraint, so the action becomes

$$S = z(T) + \int_0^T \lambda(\dot z - \mathcal{L}(x,\dot x,z))\mathrm{d}t = z(0) + \int_0^T \dot z + \lambda(\dot z - \mathcal{L}(x,\dot x,z))\mathrm{d}t$$

Vary x, z with fixed endpoints:

$$0 = \frac{\delta S}{\delta x} = \lambda \frac{\partial \mathcal{L}}{\partial x} - \lambda \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot x} - \dot\lambda \frac{\partial \mathcal{L}}{\partial \dot x}$$
$$0 = \frac{\delta S}{\delta z} = -\dot\lambda - \lambda \frac{\partial \mathcal{L}}{\partial z}$$

Hence $\dot\lambda = -\lambda \frac{\partial \mathcal{L}}{\partial z}$ $\qquad$ $\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot x} + \frac{\partial \mathcal{L}}{\partial z}\frac{\partial \mathcal{L}}{\partial \dot x} = 0$

If we restrict to solution curves, but vary the endpoints, we get $\delta S = \Big[\lambda(\delta z - p\delta x)\Big]_0^T$.

[León, Lainz, Muñoz-Lecanda. The Herglotz principle and vakonomic dynamics. 2021]

Nonholonomic (chetaev) approach

Consider $S = \int_0^T \mathcal{L}$ and impose two separate constraints:

- Dynamical constraint: $\dot{z} = \mathcal{L}$
- Constraint on the variations considered: $\delta z = \frac{\partial \mathcal{L}}{\partial \dot{x}} \delta x$

Then

$$\begin{aligned}\delta S &= \int_0^T \frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\delta \dot{x} + \frac{\partial \mathcal{L}}{\partial z}\delta z \,\mathrm{d}t \\ &= \int_0^T \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} + \frac{\partial \mathcal{L}}{\partial z}\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x \,\mathrm{d}t + \left[\frac{\partial \mathcal{L}}{\partial \dot{x}} \delta x \right]_0^T\end{aligned}$$

Warning

In generalisation to field theory, nonholonomic and vakonomic principles are not equivalent!

[Gaset, Lainz, Mas, Rivas. The Herglotz variational principle for dissipative field theories. 2024]

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Discrete Herglotz variational principle

Variational integrator:

$$\text{approximate} \quad \int_t^{t+h} \mathcal{L}(x, \dot{x}, z)\,\mathrm{d}t \quad \text{by} \quad hL(x_j, x_{j+1}, z_j; h),$$

where $h > 0$ is the step size.

Discrete Herglotz variational principle

Given $x = (x_0, \ldots, x_N) \in Q^{N+1}$ we define $z = (z_0, \ldots, z_N) \in \mathbb{R}^{N+1}$ by $z_0 = 0$ and

$$z_{j+1} - z_j = hL(x_j, x_{j+1}, z_j; h)$$

Look for a discrete curve x such that

$$\frac{\mathrm{d}z_N}{\mathrm{d}x_j} = 0 \qquad \forall j \in \{1, \ldots, N-1\}.$$

Idea: variations of x do not affect the final value of z.

Direct (implicit) approach

Denote by D_i the partial derivative wrt i-th entry.

From

$$z_j = z_{j-1} + hL(x_{j-1}, x_j, z_{j-1})$$

it follows that variations wrt x_i evolve as

$$\begin{aligned}\frac{\partial z_j}{\partial x_i} &= \frac{\partial z_{j-1}}{\partial x_i} + h\mathrm{D}_1 L(x_{j-1}, x_j, z_{j-1})\frac{\partial x_{j-1}}{\partial x_i} + h\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1})\frac{\partial x_j}{\partial x_i} + h\mathrm{D}_3 L(x_{j-1}, x_j, z_{j-1})\frac{\partial z_{j-1}}{\partial x_i} \\ &= (1 + h\mathrm{D}_3 L(x_{j-1}, x_j, z_{j-1}))\frac{\partial z_{j-1}}{\partial x_i} + h\mathrm{D}_1 L(x_{j-1}, x_j, z_{j-1})\delta^i_{j-1} + h\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1})\delta^i_j\end{aligned}$$

This implies that

Lemma

For h sufficiently small: $\displaystyle\frac{\partial z_N}{\partial x_i} = 0 \quad \forall i \qquad \Leftrightarrow \qquad \frac{\partial z_{i+1}}{\partial x_i} = 0 \quad \forall i$

Direct (implicit) approach

$z_i = z_{i-1} + hL(x_{i-1}, x_i, z_{i-1})$ implies

$$\frac{\partial z_i}{\partial x_i} = h\mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1})$$

$z_{i+1} = z_i + hL(x_i, x_{i+1}, z_i)$ implies

$$\begin{aligned}\frac{\partial z_{i+1}}{\partial x_i} &= \frac{\partial z_i}{\partial x_i} + h\mathrm{D}_1 L(x_i, x_{i+1}, z_i) + h\mathrm{D}_3 hL(x_i, x_{i+1}, z_i)\frac{\partial z_i}{\partial x_i} \\ &= h\mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) + h\mathrm{D}_1 L(x_i, x_{i+1}, z_i) + h^2\mathrm{D}_3 hL(x_i, x_{i+1}, z_i)\mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1})\end{aligned}$$

Discrete generalized Euler-Lagrange equation

$$0 = \mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}) + \mathrm{D}_1 L(x_j, x_{j+1}, z_j) + h\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1})\mathrm{D}_3 L(x_j, x_{j+1}, z_j).$$

where D_i is the partial derivative w.r.t. the i-th variable.

Discrete Herglotz equations

Discrete generalized Euler-Lagrange equation

$$0 = \mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}) + \mathrm{D}_1 L(x_j, x_{j+1}, z_j) + h\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1})\mathrm{D}_3 L(x_j, x_{j+1}, z_j).$$

If L a consistent discretization of a continuous Lagrangian $\mathcal{L}$,

$$hL \approx \int_t^{t+H} \mathcal{L} \qquad \text{so} \qquad L \approx \mathcal{L}$$

$$\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}) + \mathrm{D}_1 L(x_j, x_{j+1}, z_j) \approx \frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$h\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}) \approx \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$\mathrm{D}_3 L(x_j, x_{j+1}, z_j) \approx \frac{\partial \mathcal{L}}{\partial z}$$

Allowing dependence on z_j and z_{j+1}

Discrete generalized Euler-Lagrange equation

$$0 = \mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}, z_j) + \mathrm{D}_1 L(x_j, x_{j+1}, z_j, z_{j+1}) \\ + \tfrac{h\mathrm{D}_2 L(x_{j-1},x_j,z_{j-1},z_j)}{1-h\mathrm{D}_4 L(x_{j-1},x_j,z_{j-1},z_j)} \big(\mathrm{D}_3 L(x_j, x_{j+1}, z_j, z_{j+1}) + \mathrm{D}_4 L(x_{j-1}, x_j, z_{j-1}, z_j)\big).$$

where D_i is the partial derivative w.r.t. the i-th variable.

If L a consistent discretization of a continuous Lagrangian $\mathcal{L}$,

$$\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}, z_j) + \mathrm{D}_1 L(x_j, x_{j+1}, z_j, z_{j+1}) \approx \frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$\frac{h\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}, z_j)}{1 - h\mathrm{D}_4 L(x_{j-1}, x_j, z_{j-1}, z_j)} \approx \frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$\mathrm{D}_3 L(x_j, x_{j+1}, z_j, z_{j+1}) + \mathrm{D}_4 L(x_{j-1}, x_j, z_{j-1}, z_j) \approx \frac{\partial \mathcal{L}}{\partial z}$$

Contact structure

The discrete generalized Euler-Lagrange equation can be written as

$$\frac{h\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}, z_j)}{1 - h\mathrm{D}_4 L(x_{j-1}, x_j, z_{j-1}, z_j)} + \frac{h\mathrm{D}_1 L(x_j, x_{j+1}, z_j, z_{j+1})}{1 + h\mathrm{D}_3 L(x_j, x_{j+1}, z_j, z_{j+1})} = 0$$

Position-momentum formulation

$$F : T^*Q \times \mathbb{R} \mapsto T^*Q \times \mathbb{R} : (x_{j-1}, p_{j-1}, z_{j-1}) \mapsto (x_j, p_j, z_j),$$

where $p_j = p_j^- = p_j^+$ and

$$p_j^- = \frac{h\mathrm{D}_2 L(x_{j-1}, x_j, z_{j-1}, z_j)}{1 - h\mathrm{D}_4 L(x_{j-1}, x_j, z_{j-1}, z_j)},$$

$$p_j^+ = -\frac{h\mathrm{D}_1 L(x_j, x_{j+1}, z_j, z_{j+1})}{1 + h\mathrm{D}_3 L(x_j, x_{j+1}, z_j, z_{j+1})}.$$

The map F is a contact transformation with respect to the 1-form $\mathrm{d}z - p\,\mathrm{d}x$.

Vakonomic approach

$$
\begin{aligned}
S &= z_N + \lambda_j (z_{j+1} - z_j - hL(x_j, x_{j+1}, z_j)) \\
&= z_0 + \sum_j (z_{j+1} - z_j) + \lambda_j (z_{j+1} - z_j - hL(x_j, x_{j+1}, z_j))
\end{aligned}
$$

Vary both x and z:

$$
\begin{aligned}
0 &= \frac{\partial S}{\partial x_i} = -\lambda_i h \mathrm{D}_1 L(x_i, x_{i+1}, z_i) - \lambda_{i-1} h \mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) \\
0 &= \frac{\partial S}{\partial z_i} = \lambda_{i-1} - \lambda_i - \lambda_i h \mathrm{D}_3 L(x_i, x_{i+1}, z_i)
\end{aligned}
$$

So

$$
-\lambda_i h \mathrm{D}_1 L(x_i, x_{i+1}, z_i) - (\lambda_i + \lambda_i h \mathrm{D}_3 L(x_i, x_{i+1}, z_i)) h \mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) = 0
$$

and again we find

$$
\mathrm{D}_1 L(x_i, x_{i+1}, z_i) + \mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) + h \mathrm{D}_3 L(x_i, x_{i+1}, z_i)) \mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) = 0
$$

Nonholonmic approach

$$S = \sum_i hL(x_i, x_{i+1}, z_i)$$

Consider only those variations satisfying

$$\delta z_j = \mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) \delta x_j$$

Then

$$\begin{aligned}
\delta S &= \sum_i hD_1 L(x_i, x_{i+1}, z_i)\delta x_i + \mathrm{D}_2 L(x_i, x_{i+1}, z_i)\delta x_{i+1} + \mathrm{D}_3 L(x_i, x_{i+1}, z_i)\delta z_i \\
&= \sum_i h\big(\mathrm{D}_1 L(x_i, x_{i+1}, z_i) + \mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) + \mathrm{D}_3 L(x_i, x_{i+1}, z_i)\mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1})\big)\delta x_i
\end{aligned}$$

and again we find

$$\mathrm{D}_1 L(x_i, x_{i+1}, z_i) + \mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) + h\mathrm{D}_3 L(x_i, x_{i+1}, z_i))\mathrm{D}_2 L(x_{i-1}, x_i, z_{i-1}) = 0$$

Backward error analysis

Solutions of the difference equations

$$\begin{cases} \dfrac{z_{j+1} - z_j}{h} = L(x_j, x_{j+1}, z_j, z_{j+1}; h) \\ \dfrac{x_{j+1} - 2x_j + x_{j-1}}{h^2} = F(x_{j-1}, x_j, x_{j+1}, z_{j-1}, z_j, z_{j+1}; h). \end{cases}$$

are formally interpolated by solutions of the modified equations

$$\begin{cases} \dot{z} = \mathcal{L}_{\mathrm{mod}}(x, \dot{x}, z, h) = \mathcal{L}(x, \dot{x}, z) + h\mathcal{L}_1(x, \dot{x}, z) + h^2\mathcal{L}_2(x, \dot{x}, z) + \dots \\ \ddot{x} = f_{\mathrm{mod}}(x, \dot{x}, z; h) = f(x, \dot{x}, z) + hf_1(x, \dot{x}, z) + h^2 f_2(x, \dot{x}, z) + \dots \end{cases}.$$

(The power series are usually not convergent. Truncations need to be used to make rigorous statements.)

The modified equations are also a contact system

In particular, $\ddot{x} = f_{\mathrm{mod}}(x, \dot{x}, z; h)$ is the generalized Euler-Lagrange equation of $\mathcal{L}_{\mathrm{mod}}(x, \dot{x}, z, h)$.

Conclusions

- There are (at least) three implementations of the Herglotz variational principle:
  - direct (implicit)
  - vakonomic
  - nonholonomic (Chetaev)

  In mechanics (discrete/continuous), they are all equivalent.
- Options for discrete Lagrangians
  - $L(x_j, x_{j+1}, z_j)$: more explicit difference equations
  - $L(x_j, x_{j+1}, z_j, z_{j+1})$: more symmetric difference equations
- Hamiltonian (splitting) integrators are also available for contact Hamiltonian systems.
- Structure-preserving discretizations for contact systems can be obtained using many of the same ideas as for symplectic systems.

  But relevance of long-time conservation properties is not as obvious.

Selected references

References: Herglotz principle

[Herglotz. Beruhrungstransformationen. University of Gottingen, 1930.]

[Georgieva, Guenther, Bodurov. Generalized variational principle of Herglotz for several independent variables. First Noether-type theorem. Journal of Mathematical Physics, 2003]

[León, Lainz, Muñoz-Lecanda. The Herglotz principle and vakonomic dynamics. Geometric Science of Information, 2021]

[Gaset, Lainz, Mas, Rivas. The Herglotz variational principle for dissipative field theories. Geometric Mechanics, 2024]

References: discretisation

[V, Bravetti, Seri. Contact variational integrators. J Phys A, 2019.]

[Bravetti, Seri, V, Zadra. Numerical integration in celestial mechanics: a case for contact geometry. Celestial Mechanics and Dynamical Astronomy, 2020.]

[Bravetti, Seri, Zadra. New directions for contact integrators. International Conference on Geometric Science of Information, 2021]

[Anahory Simoes, Martín de Diego, Lainz Valcázar, de León. On the Geometry of Discrete Contact Mechanics. J Nonlinear Sci, 2021]