Lagrangian multiforms and bi-Hamiltonian systems

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Contents

Motivation

Lagrangian multiform theory (aka pluri-Lagrangian systems)

Variational principle for hierarchies of PDEs.
First step to construct: find a Lagrangian for one member of the hierarchy.

Example: KdV equation $u_t = u_{xxx} + 3uu_x$

- Traditional EL equations do not produce scalar evolutionary equations.
- Could try differentiating:

  $$u_{xt} = u_{xxx} + 3uu_{xx} + 3u_x^2$$

  but still not an EL equation
- Need to pass to potential variable, $u = \bar{u}_x$:

  $$\bar{u}_{xt} = \bar{u}_{xxxx} + 3\bar{u}_x\bar{u}_{xx}$$

  is EL equation of $\mathcal{L} = \frac{1}{2}\bar{u}_t\bar{u}_x - \frac{1}{2}\bar{u}_x\bar{u}_{xxx} - \bar{u}_x^3$.

Motivation

Why is $\bar{u}$ the right variable?
Are there other ways to make a Hamiltonian PDE Lagrangian?

Dorfman. Dirac structures of integrable evolution equations. 1987.

Dirac structures generalise both Hamiltonian and symplectic operators, which are key ingredients in the Hamiltonian resp. Lagrangian formulation.

Mokhov. Symplectic and Poisson structures on loop spaces of smooth manifolds, and integrable systems. 1998.

Studies relations between invertible symplectic operators and Hamiltonian operators, considers classification problems, gives examples of bi-Lagrangian structures.

Nutku & Pavlov. Multi-Lagrangians for integrable systems. 2002.

Many examples of integrable PDEs with several non-equivalent Lagrangians, but the strategy to obtain them is not clearly explained.

Bustamante & Hojman. 2003. Pavlov, Vitolo. 2017.

Hamiltonian systems

In mechanics: $i_X \boldsymbol{\omega} = \mathrm{d}H$

Symplectic form ω defines an operator $\Omega : TQ \to T^*Q$

$$\Omega X = \mathrm{d}H \qquad \text{(symplectic)}$$

Non-degeneracy of ω implies that Ω is invertible, $A = \Omega^{-1} : T^*Q \to TQ$

$$X = A\mathrm{d}H \qquad \text{(Poisson)}$$

Hamiltonian systems

In mechanics: $i_X \boldsymbol{\omega} = \mathrm{d}H$

Symplectic form ω defines an operator $\Omega : TQ \to T^*Q$

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$$X = A\mathrm{d}H \qquad \text{(Poisson)}$$

In classical field theory: (Poisson) generalises to

$$u_t = \mathcal{A}\frac{\delta H}{\delta u}$$

$\mathcal{A}$ may not be invertible, so cannot write analogue of (symplectic).

Example: KdV can be written as

$$u_t = \partial_x(u_{xx} + \tfrac{3}{2}u^2) = \mathcal{A}\frac{\delta H}{\delta u}$$

with $\mathcal{A} = \partial_x$ and $H = -\frac{1}{2}u_x^2 + \frac{1}{2}u^3$.

Lagrangian mechanics in phase space

Lagrangian description of

$$\Omega z_t = \nabla H(z)$$

Variational principle in phase space:

$$\mathcal{L} = \frac{1}{2} \boldsymbol{z}^\top \Omega \boldsymbol{z}_t - H(\boldsymbol{z})$$

In Darboux coordinates,

$$\mathcal{L} = \sum_i q^i p_i - H$$

$(\boldsymbol{z} : \mathbb{R} \to T^* Q)$

Lagrangian description of

$$z_t = A \nabla H(z)$$

Introduce new variables $\bar{\boldsymbol{z}}$ by

$$\boldsymbol{z} = A \bar{\boldsymbol{z}}$$

If A is constant: $\nabla_{\bar{\boldsymbol{z}}} = A^\top \nabla_{\boldsymbol{z}}$.

If A is skew-symmetric:

$$\begin{aligned} A \bar{\boldsymbol{z}}_t &= A \nabla_{\boldsymbol{z}} H(\boldsymbol{z}) \\ &= -\nabla_{\bar{\boldsymbol{z}}} (H(A \bar{\boldsymbol{z}})) \end{aligned}$$

which is the EL equation of

$$\mathcal{L} = \tfrac{1}{2} \bar{\boldsymbol{z}}^\top A \bar{\boldsymbol{z}}_t + H(\bar{\boldsymbol{z}})$$

Does this generalise to Hamiltonian PDEs $u_t = \mathcal{A} \dfrac{\delta H}{\delta u}$?

Contents

Potential variable as dual space variable

- Space of independent (space-)variables $\mathbb{R}^m$.
- Dependent variables take values in vector space U.
- Dual space $\bar{U}$, bilinear pairing $\langle \cdot, \cdot \rangle : U \times \bar{U} \to \mathbb{R}$.
- Phase space $\mathcal{F} = \{\mathbb{R}^m \to U \mid \text{smooth, rapidly decreasing}\}$,
  and dual $\bar{\mathcal{F}} = \{\mathbb{R}^m \to \bar{U} \mid \text{smooth, rapidly decreasing}\}$.
- Pairing extends to $$\begin{aligned} \langle \cdot, \cdot \rangle &: \mathcal{F} \times \bar{\mathcal{F}} \to C^\infty(\mathbb{R}^m, \mathbb{R}). \\ \textstyle\int \langle \cdot, \cdot \rangle \mathrm{d}^m x &: \mathcal{F} \times \bar{\mathcal{F}} \to \mathbb{R}. \end{aligned}$$

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- Pairing extends to $\langle\cdot,\cdot\rangle : \mathcal{F} \times \bar{\mathcal{F}} \to C^\infty(\mathbb{R}^m, \mathbb{R})$.
  $\int \langle\cdot,\cdot\rangle \mathrm{d}^m x : \mathcal{F} \times \bar{\mathcal{F}} \to \mathbb{R}$.

Fibre of $T\mathcal{F}$ can be identified with $\mathcal{F}$, fibre of $T^*\mathcal{F}$ with $\bar{\mathcal{F}}$.

Hamiltonian operator on $\mathcal{F}$: fibre-preserving map $\mathcal{A} : \Gamma(T^*\mathcal{F}) \to \Gamma(T\mathcal{F})$ from 1-forms to vector fields, satisfying certain conditions.

Constant Hamiltonian operator: $\mathcal{A} : \bar{\mathcal{F}} \to \mathcal{F}$.

Potential Hamiltonian variable $\bar{u}$, defined by $u = \mathcal{A}\bar{u}$

The phase space is now $\bar{\mathcal{F}}$. $\bar{U}$ becomes the primary space and U the dual.

Hamiltonian and symplectic operators

$\mathcal{A} : \bar{\mathcal{F}} \to \mathcal{F}$ is a constant Hamiltonian operator on $\mathcal{F}$ if

- $\mathcal{A}$ is skew-adjoint: $\int \langle f, \mathcal{A} g \rangle \mathrm{d}^m x = - \int \langle \mathcal{A} f, g \rangle \mathrm{d}^m x$, where $f, g \in \bar{\mathcal{F}}$
- $\{F, G\}_{\mathcal{F}} := \int \langle \frac{\delta F}{\delta u}, \mathcal{A} \frac{\delta G}{\delta u} \rangle \mathrm{d}^m x$ satisfies the Jacobi identity,
  where $F, G : \mathcal{F} \to \mathbb{R}$ and $\frac{\delta F}{\delta u}, \frac{\delta G}{\delta u} : \mathcal{F} \to \bar{\mathcal{F}}$

$\mathcal{J} : \mathcal{F} \to \bar{\mathcal{F}}$ is a constant symplectic operator on $\mathcal{F}$ if

- $\mathcal{J}$ is skew-adjoint: $\int \langle X, \mathcal{J} Y \rangle \mathrm{d}^m x = - \int \langle \mathcal{J} X, Y \rangle \mathrm{d}^m x$,
- $\omega(X, Y) := \int \langle X, \mathcal{J} Y \rangle \mathrm{d}^m x$ defines a closed 2-form,
  where $X, Y : \mathcal{F} \to \mathcal{F}$

The following are equivalent:

- $\mathcal{J}$ is symplectic
- there exists a $p[u] = p(u, u_x, u_{xx}, \dots)$ such that $\mathcal{J} = \ell_p^* - \ell_p$
- $\mathcal{J} u_t$ is the Euler-Lagrange expression of $\mathcal{L} = p[u] u_t$

In particular, $\mathcal{J} u_t = -\dfrac{\delta H}{\delta u}$ is an Euler-Lagrange equation if $\mathcal{J}$ is symplectic.

Relating Hamiltonian and symplectic operators

Theorem [e.g. Mokhov, 1998]

$\mathcal{A}$ is a Hamiltonian operator if and only if $\mathcal{A}^{-1}$ is a symplectic operator

Proof idea: Find vectors X, Y, Z and functionals F, G, H such that $\mathrm{d}\omega(X, Y, Z) = \{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\}$ ■

Lemma

If $\mathcal{A} : \bar{\mathcal{F}} \to \mathcal{F}$ is a constant Hamiltonian operator on $\mathcal{F}$ and $H : \mathcal{F} \to \mathbb{R}$, then

$$\frac{\delta H \circ \mathcal{A}}{\delta \bar{u}} = \mathcal{A}^* \frac{\delta H}{\delta u}$$

Relating Hamiltonian and symplectic operators

Theorem [e.g. Mokhov, 1998]

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Lemma

If $\mathcal{A} : \bar{\mathcal{F}} \to \mathcal{F}$ is a constant Hamiltonian operator on $\mathcal{F}$ and $H : \mathcal{F} \to \mathbb{R}$, then

$$\frac{\delta H \circ \mathcal{A}}{\delta \bar{u}} = \mathcal{A}^* \frac{\delta H}{\delta u} = -\mathcal{A} \frac{\delta H}{\delta u}$$

If $\mathcal{A}$ is constant and invertible, the Lemma allows us to identify the bracket

$$\{F,G\}_{\bar{\mathcal{F}}} := \int \left\langle \frac{\delta F}{\delta \bar{u}}, \mathcal{A}^{-1} \frac{\delta G}{\delta \bar{u}} \right\rangle \mathrm{d}^m x$$

with the Poisson bracket $-\{\cdot,\cdot\}_{\mathcal{F}}$, so $\mathcal{A}^{-1}$ is a Hamiltonian operator on $\bar{\mathcal{F}}$
and $\mathcal{A}$ is a symplectic operator on $\bar{\mathcal{F}}$

Express $u_t = \mathcal{A}\frac{\delta H}{\delta u}$ wrt potential Hamiltonian variable

Using $u = \mathcal{A}\bar{u}$ and the Lemma:

$$u_t = \mathcal{A}\frac{\delta H[u]}{\delta u} \qquad \Longrightarrow \qquad \mathcal{A}\bar{u}_t = -\frac{\delta H[\mathcal{A}\bar{u}]}{\delta \bar{u}}$$

Since $\mathcal{A}$ is a symplectic operator on $\bar{\mathcal{F}}$, the is the EL equation of a Lagrangian of the form

$$\mathcal{L} = p[\bar{u}]\bar{u}_t - H[\mathcal{A}\bar{u}]$$

The same conclusion holds if $\mathcal{A}$ is only surjective (and constant).

Switching to the potential Hamiltonian variable $\bar{u}$

- switches the roles of $\mathcal{F}$ and $\bar{\mathcal{F}}$,
- turns the Hamiltonian operator $\mathcal{A}$ into a symplectic operator,
- turns a Hamiltonian PDE into an EL equation.

Contents

Example: KdV

$u_t = u_{xxx} + 3uu_x$ can be written as

$$u_t = \partial_x(u_{xx} + \tfrac{3}{2}u^2) \qquad\qquad = \partial_x \frac{\delta}{\delta u}(-\tfrac{1}{2}u_x^2 + \tfrac{1}{2}u^3)$$

$$u_t = (\partial_x^3 + 2u\partial_x + u_x)(u) \qquad\qquad = (\partial_x^3 + 2u\partial_x + u_x)\frac{\delta}{\delta u}(\tfrac{1}{2}u^2)$$

Use $\mathcal{A} = \partial_x$ to define potential variable, $u = \bar{u}_x$.

Not just $\mathcal{A}$, but also $\mathcal{B} = \partial_x^3 + 2\bar{u}_x\partial_x + \bar{u}_x$ is symplectic in $\bar{u}$-variables.

We have two Lagrangians:

$$\begin{aligned} \mathcal{L}_{\mathcal{A}} &= \tfrac{1}{2}\bar{u}_x\bar{u}_t & &- (-\tfrac{1}{2}\bar{u}_{xx}^2 + \tfrac{1}{2}\bar{u}_x^3) \\ \mathcal{L}_{\mathcal{B}} &= \tfrac{1}{2}\left(\bar{u}_{xxx} + \bar{u}_x^2\right)\bar{u}_t & &- (\tfrac{5}{8}\bar{u}_x^4 - \tfrac{5}{2}\bar{u}_x\bar{u}_{xx}^2 + \tfrac{1}{2}\bar{u}_{xxx}^2) \end{aligned}$$

with EL equations

$$\mathcal{A}(-\bar{u}_t + \bar{u}_{xxx} + \tfrac{3}{2}\bar{u}_x^2) = 0$$

$$\mathcal{B}(-\bar{u}_t + \bar{u}_{xxx} + \tfrac{3}{2}\bar{u}_x^2) = 0.$$

Double Lenard scheme

$\mathcal{A}, \mathcal{B}$ Hamiltonian pair, $\mathcal{A}$ constant,

$u = \mathcal{A}\bar{u}$,

h_{-1} a Casimir of $\mathcal{A}$.

$$\begin{tikzcd}
\frac{\delta h_{-1}}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & 0 & & \\
\frac{\delta h_0}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & u_{t_0} & \bar{u}_{t_0} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_0}{\delta \bar{u}} \\
\frac{\delta h_1}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & u_{t_1} & \bar{u}_{t_1} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_1}{\delta \bar{u}} \\
\frac{\delta h_2}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & u_{t_2} & \bar{u}_{t_2} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_2}{\delta \bar{u}} \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{Hamiltonian Lenard Scheme}}$ $\underbrace{\qquad\qquad\qquad\qquad}_{\text{Symplectic Lenard Scheme}}$

Hamiltonian recursion

$$\begin{tikzcd}
\frac{\delta h_{-1}}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & 0 & & \\
\frac{\delta h_0}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] \arrow[ddr, magenta, "\mathcal{B}\mathcal{A}^{-1}\mathcal{B}"] \arrow[dddr, magenta, "\mathcal{B}\mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1}\mathcal{B}"'] & u_{t_0} & \bar{u}_{t_0} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_0}{\delta \bar{u}} \\
\frac{\delta h_1}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & u_{t_1} & \bar{u}_{t_1} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_1}{\delta \bar{u}} \\
\frac{\delta h_2}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & u_{t_2} & \bar{u}_{t_2} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_2}{\delta \bar{u}} \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

Hierarchy of Hamiltonian operators obtained by recursion operator $\mathcal{B}\mathcal{A}^{-1}$:

$$\mathcal{A},\ \mathcal{B},\ \mathcal{B}\mathcal{A}^{-1}\mathcal{B},\ \mathcal{B}\mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1}\mathcal{B},\ \ldots$$

Hamiltonian recursion

$$\begin{tikzcd}
\frac{\delta h_{-1}}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & 0 & & \\
\frac{\delta h_0}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] \arrow[ur, magenta, "\mathcal{A}\mathcal{B}^{-1}\mathcal{A}"] \arrow[ddr, magenta, "\mathcal{B}\mathcal{A}^{-1}\mathcal{B}"] \arrow[dddr, magenta, "\mathcal{B}\mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1}\mathcal{B}"'] & u_{t_0} & \bar{u}_{t_0} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_0}{\delta \bar{u}} \\
\frac{\delta h_1}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & u_{t_1} & \bar{u}_{t_1} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_1}{\delta \bar{u}} \\
\frac{\delta h_2}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & u_{t_2} & \bar{u}_{t_2} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_2}{\delta \bar{u}} \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

Hierarchy of Hamiltonian operators obtained by recursion operator $\mathcal{B}\mathcal{A}^{-1}$:

$$\ldots,\ \mathcal{A}\mathcal{B}^{-1}\mathcal{A}, \qquad \mathcal{A},\ \mathcal{B},\ \mathcal{B}\mathcal{A}^{-1}\mathcal{B},\ \mathcal{B}\mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1}\mathcal{B},\ \ldots$$

Hamiltonian operators on $\mathcal{F}$ and $\bar{\mathcal{F}}$

Lemma

If $\mathcal{D}$ is a Hamiltonian operator on $\mathcal{F}$ (i.e. wrt u),
then $\mathcal{A}^{-1}\mathcal{D}\mathcal{A}^{-1}$ is a Hamiltonian operator on $\bar{\mathcal{F}}$ (i.e. wrt $\bar{u}$)

Proof. Recall that $\frac{\delta F}{\delta \bar{u}} = -\mathcal{A}\frac{\delta F}{\delta u}$ so

$$\{F, G\} = \int \frac{\delta F}{\delta u}\mathcal{D}\frac{\delta G}{\delta u}\mathrm{d}^m x = -\int \frac{\delta F}{\delta \bar{u}}\mathcal{A}^{-1}\mathcal{D}\mathcal{A}^{-1}\frac{\delta G}{\delta \bar{u}}\mathrm{d}^m x \qquad \blacksquare$$

$$\begin{array}{llll}
\mathcal{D} = \mathcal{A} & \rightarrow & \mathcal{A}^{-1} \text{ Hamiltonian for } \bar{u}, & \mathcal{A} \text{ symplectic for } \bar{u} \\
\mathcal{D} = \mathcal{B} & \rightarrow & \mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1} \text{ Hamiltonian for } \bar{u}, & \mathcal{A}\mathcal{B}^{-1}\mathcal{A} \text{ symplectic for } \bar{u} \\
\mathcal{D} = \mathcal{A}\mathcal{B}^{-1}\mathcal{A} & \rightarrow & \mathcal{B}^{-1} \text{ Hamiltonian for } \bar{u}, & \mathcal{B} \text{ symplectic for } \bar{u}
\end{array}$$

$$\begin{tikzcd}
\frac{\delta h_i}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & u_{t_i} & \bar{u}_{t_i} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_i}{\delta \bar{u}} \\
\frac{\delta h_{i+1}}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[ur, "\mathcal{A}\mathcal{B}^{-1}\mathcal{A}"] & u_{t_{i+1}} & \bar{u}_{t_{i+1}} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] & -\frac{\delta h_{i+1}}{\delta \bar{u}}
\end{tikzcd}$$

Dual picture

$$\begin{tikzcd}
\frac{\delta h_i}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & u_{t_i} & \bar{u}_{t_i} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_i}{\delta \bar{u}} \\
\frac{\delta h_{i+1}}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[ur, "\mathcal{A}\mathcal{B}^{-1}\mathcal{A}"] & u_{t_{i+1}} & \bar{u}_{t_{i+1}} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] & -\frac{\delta h_{i+1}}{\delta \bar{u}}
\end{tikzcd}$$

- $\mathcal{A} : T\bar{\mathcal{F}} \to T^*\bar{\mathcal{F}}$ is symplectic on $\bar{\mathcal{F}}$ and dual to $\mathcal{A} : T^*\mathcal{F} \to T\mathcal{F}$:

$$u_{t_i} = \mathcal{A}\frac{\delta h_i}{\delta u} \qquad \text{and} \qquad \mathcal{A}\bar{u}_{t_i} = \frac{\delta h_i}{\delta \bar{u}}$$

- $\mathcal{B} : T\bar{\mathcal{F}} \to T^*\bar{\mathcal{F}}$ is symplectic on $\bar{\mathcal{F}}$ but dual to $\mathcal{A}\mathcal{B}^{-1}\mathcal{A}$, not to $\mathcal{B}$:

$$u_{t_i} = \mathcal{B}\frac{\delta h_{i-1}}{\delta u} \qquad \text{but} \qquad \mathcal{B}\bar{u}_{t_i} = \frac{\delta h_{i+1}}{\delta \bar{u}}$$

- Could also consider $\mathcal{B}\mathcal{A}^{-1}\mathcal{B}$, ..., as symplectic operators for $\bar{u}$.

Each of these leads to different Lagrangians in $\bar{u}$-variables.

Contents

Implications in Lagrangian multiform theory

Lagrangian multiforms for 2d PDEs

Considere functions $\bar{u} = \bar{u}(t_0 = x, t_1, t_2, \ldots, t_n)$ of multi-time.

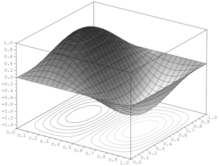
Jet-dependent 2-form $\mathcal{L} = \sum_{i<j} L_{ij}[\bar{u}] \, \mathrm{d}t^i \wedge \mathrm{d}t^j$.

For every surface $\Gamma \in \mathbb{R}^{n+1}$, consider the action

$$S_\Gamma[\bar{u}] = \int_\Gamma \mathcal{L}[\bar{u}]$$

and require that all these actions are critical with respect to variations of both Γ and u.

$\Gamma \subset \mathbb{R}^{n+1}$



$\mathbb{R}^{n+1} \xrightarrow{\bar{u}} \bar{U}$

Particular case: take Γ to be the (t_i, t_j)-plane, then $S_\Gamma = \int L_{ij}[\bar{u}] \, \mathrm{d}t^i \wedge \mathrm{d}t^j$

Variational derivatives

Notation:

- $\partial_i = \frac{\partial}{\partial t_i}$ and $u_{t_i} = \partial_i u$.
- $\bar{u}_I$ mixed partial derivative wrt the t-variables in multi-index I.
- $\bar{u}_{I t_i^\alpha} = \partial_i^\alpha u_I$

Variational derivative with respect to $\bar{u}_I$ in the direction of t_i, t_j:

$$\frac{\delta_{ij}}{\delta \bar{u}_I} := \sum_{\alpha,\beta \geq 0} (-1)^{\alpha+\beta} \partial_i^\alpha \partial_j^\beta \frac{\partial}{\partial \bar{u}_{I t_i^\alpha t_j^\beta}}$$

Variational derivative with respect to $\bar{u}_I$ in the direction of t_i, t_j, t_k:

$$\frac{\delta_{ijk}}{\delta \bar{u}_I} := \sum_{\alpha,\beta,\gamma \geq 0} (-1)^{\alpha+\beta+\gamma} \partial_i^\alpha \partial_j^\beta \partial_k^\gamma \frac{\partial}{\partial \bar{u}_{I t_i^\alpha t_j^\beta t_k^\gamma}}.$$

Theorem: multiform Euler-Lagrange equations

Let $\mathrm{d}\mathcal{L}[\bar{u}] = \sum_{i<j<k} P_{ijk}[\bar{u}]\,\mathrm{d}t_i \wedge \mathrm{d}t_j \wedge \mathrm{d}t_k$. The following are equivalent:

1. $\bar{u} : \mathbb{R}^N \to \mathbb{R}$ satisfies the pluri-Lagrangian principle
2. "$\delta \mathrm{d}\mathcal{L} = 0$" i.e. for all i, j, k, there holds $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} P_{ijk}[\bar{u} + \varepsilon \bar{v}]\big|_{\varepsilon=0} = 0$
3. $\frac{\partial}{\partial \bar{u}_I} P_{ijk}[\bar{u}] = 0$
4. $\frac{\delta_{ijk}}{\delta \bar{u}_I} P_{ijk}[\bar{u}] = 0$
5. $\bar{u} : \mathbb{R}^N \to \mathbb{R}$ satisfies the system of equations

$$\frac{\delta_{ij} L_{ij}}{\delta \bar{u}_I} = 0 \qquad \forall I \not\ni t_i, t_j \tag{1}$$

$$\frac{\delta_{ij} L_{ij}}{\delta \bar{u}_{It_i}} + \frac{\delta_{jk} L_{jk}}{\delta \bar{u}_{It_k}} = 0 \qquad \forall I \not\ni t_j \tag{2}$$

$$\frac{\delta_{ij} L_{ij}}{\delta \bar{u}_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta \bar{u}_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta \bar{u}_{It_k t_i}} = 0 \qquad \forall I \tag{3}$$

Double zero property

Theorem

Consider a system of equations $E[\bar{u}] = 0$ with $E[\bar{u}] = (E_1[\bar{u}], \ldots, E_n[\bar{u}])^\top$. If

$$P_{ijk}[\bar{u}] = (\mathcal{F}_{ijk}[\bar{u}] E[\bar{u}]) \cdot (\mathcal{G}_{ijk}[\bar{u}] E[\bar{u}]),$$

where $\mathcal{F}_{ijk}[\bar{u}]$ and $\mathcal{G}_{ijk}[\bar{u}]$ are $(m \times n)$-matrix differential operators for some m, then the Lagrangian multiform principle is satisfied for all functions $\bar{u}$ that solve the system of equations $E[\bar{u}] = 0$.

Proof. If $\bar{u}$ satisfies $E[\bar{u}] = 0$, then it also satisfies

$$\mathcal{F}_{ijk}[\bar{u}] E[\bar{u}] = 0 \quad \text{and} \quad \mathcal{G}_{ijk}[\bar{u}] E[\bar{u}] = 0.$$

Hence $P_{ijk}[\bar{u}] = 0$ and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} P_{ijk}[\bar{u} + \varepsilon \bar{v}] \Big|_{\varepsilon=0} = \Big(& \frac{\mathrm{d}}{\mathrm{d}\varepsilon} (\mathcal{F}_{ijk}[\bar{u}] E[\bar{u}]) \Big|_{\varepsilon=0} \cdot (\mathcal{G}_{ijk}[\bar{u}] E[\bar{u}]) \\ & + (\mathcal{F}_{ijk}[\bar{u}] E[\bar{u}]) \cdot \frac{\mathrm{d}}{\mathrm{d}\varepsilon} (\mathcal{G}_{ijk}[\bar{u}] E[\bar{u}]) \Big|_{\varepsilon=0} \Big) = 0 \end{aligned}$$

■

Constructing a multiform for a given symplectic operator

Suppose we have a symplectic operator $\mathcal{A} = \ell_p - \ell_p^*$ and a hierarchy of Hamiltonians $h_1, h_2, \ldots$. Define

$$L_{0j} = p[\bar{u}]\bar{u}_{t_j} - h_j[\bar{u}] \qquad \text{where } [\bar{u}] = (\bar{u}, \bar{u}_x, \bar{u}_{xx}, \ldots)$$

Then

$$\begin{aligned} \frac{\delta_{0j} L_{0j}}{\delta \bar{u}} &= \sum_{\alpha \geq 0} (-\partial_x)^\alpha \left(\frac{\partial p[\bar{u}]}{\partial \bar{u}_{x^\alpha}} \bar{u}_{t_j} \right) - \partial_{t_j} p[\bar{u}] - \frac{\delta_{0j} h_j[\bar{u}]}{\delta \bar{u}} \\ &= \ell_p^* \bar{u}_{t_j} - \ell_p \bar{u}_{t_j} - \frac{\delta_{0j} h_j}{\delta \bar{u}} \end{aligned}$$

so the EL equations are

$$\mathcal{A} \bar{u}_{t_j} = - \frac{\delta_{0j} h_j}{\delta \bar{u}}$$

Assume they can be written as $\mathcal{A}(\bar{u}_{t_j} - Q_j)$ and consider the Poisson bracket

$$\{h_i, h_j\} = \int \frac{\delta h_i}{\delta \bar{u}} \mathcal{A}^{-1} \frac{\delta h_j}{\delta \bar{u}} \mathrm{d}x = - \int Q_i \mathcal{A} Q_j \mathrm{d}x = 0$$

Constructing a multiform for a given symplectic operator

Let $\sim$ denote equality modulo adding a total x-derivative. $L_{0j} = p\bar{u}_{t_j} - h_j$.

Aiming to have a double-zero expression for $\mathrm{d}\mathcal{L}$, we compute:

$$
\begin{aligned}
&\partial_{t_k} L_{0j} - \partial_{t_j} L_{0k} \\
&= (\ell_p \bar{u}_{t_k})\bar{u}_{t_j} - (\ell_p \bar{u}_{t_j})\bar{u}_{t_k} - (\partial_{t_k} h_j) + (\partial_{t_j} h_k) \\
&\sim \tfrac{1}{2}(\ell_p \bar{u}_{t_k})\bar{u}_{t_j} + \tfrac{1}{2} u_{t_k}(\ell_p^* \bar{u}_{t_j}) - \tfrac{1}{2}(\ell_p \bar{u}_{t_j})\bar{u}_{t_k} - \tfrac{1}{2}\bar{u}_{t_j}(\ell_p^* \bar{u}_{t_k}) - \bar{u}_{t_k}\frac{\delta h_j}{\delta \bar{u}} + \bar{u}_{t_j}\frac{\delta h_k}{\delta \bar{u}} \\
&= \tfrac{1}{2}\bar{u}_{t_j}\mathcal{A}\bar{u}_{t_k} - \tfrac{1}{2}\bar{u}_{t_k}\mathcal{A}\bar{u}_{t_j} + \bar{u}_{t_k}\mathcal{A}Q_j - \bar{u}_{t_j}\mathcal{A}Q_k \\
&\sim \tfrac{1}{2}\bar{u}_{t_j}\mathcal{A}\bar{u}_{t_k} - \tfrac{1}{2}\bar{u}_{t_k}\mathcal{A}\bar{u}_{t_j} + \tfrac{1}{2}\bar{u}_{t_k}\mathcal{A}Q_j - \tfrac{1}{2}Q_j\mathcal{A}\bar{u}_{t_k} - \tfrac{1}{2}\bar{u}_{t_j}\mathcal{A}Q_k + \tfrac{1}{2}Q_k\mathcal{A}\bar{u}_{t_j} \\
&= \tfrac{1}{2}(\bar{u}_{t_j} - Q_j)\mathcal{A}(\bar{u}_{t_k} - Q_k) - \tfrac{1}{2}(\bar{u}_{t_k} - Q_k)\mathcal{A}(\bar{u}_{t_j} - Q_j) - \tfrac{1}{2}Q_j\mathcal{A}Q_k + \tfrac{1}{2}Q_k\mathcal{A}Q_j \\
&\sim \tfrac{1}{2}(\bar{u}_{t_j} - Q_j)\mathcal{A}(\bar{u}_{t_k} - Q_k) - \tfrac{1}{2}(\bar{u}_{t_k} - Q_k)\mathcal{A}(\bar{u}_{t_j} - Q_j) + \{h_j, h_k\}
\end{aligned}
$$

If $\{h_j, h_k\}$, it follows that there exists a function L_{jk} such that

$$P_{0jk} = \partial_k L_{0j} - \partial_j L_{0k} + \partial_0 L_{jk} = \tfrac{1}{2}(\bar{u}_{t_j} - Q_j)\mathcal{A}(\bar{u}_{t_k} - Q_k) - \tfrac{1}{2}(\bar{u}_{t_k} - Q_k)\mathcal{A}(\bar{u}_{t_j} - Q_j)$$

This defines the remaining coefficients of $\mathcal{L} = \sum_{i<j} L_{ij}\mathrm{d}t^i \wedge \mathrm{d}t^j$.

Multiform EL equations are evolutionary

Theorem

Assume the differential operator $\mathcal{A}$ *has constant order. Then* $\bar{u}$ *satisfies the Lagrangian multiform principle for* $\mathcal{L} = \sum_{i<j} L_{ij} \mathrm{d}t^i \wedge \mathrm{d}t^j$ *if and only if*

$$\bar{u}_{t_j} = Q_j[\bar{u}] \qquad \forall j$$

Proof. $P_{0jk} = \frac{1}{2}(\bar{u}_{t_j} - Q_j)\mathcal{A}(\bar{u}_{t_k} - Q_k) - \frac{1}{2}(\bar{u}_{t_k} - Q_k)\mathcal{A}(\bar{u}_{t_j} - Q_j)$ satisfies the double zero on the system of equations $\bar{u}_{t_j} = Q_j[\bar{u}]$.

For the other coefficients of $\mathrm{d}\mathcal{L}$, we find

$$\partial_x P_{ijk} = \partial_i P_{0jk} + \partial_j P_{0ki} + \partial_k P_{0ij},$$

so $\partial_x P_{ijk}$ and hence P_{ijk} also have the double zero property.

So equations $\bar{u}_{t_j} = Q_j[\bar{u}]$ imply that $\bar{u}$ satisfies the multiform principle.

Now assume $\bar{u}$ satisfies the Lagrangian multiform principle. Let $\mathcal{A} = a\partial_x^n + l.o.t.$, where $a[\bar{u}]$ is non-vanishing. Then

$$0 = \frac{\partial P_{0jk}}{\partial u_{x^n t_k}} = \frac{1}{2} a(\bar{u}_{t_j} - Q_j).$$

■

Example: KdV

Consider $\mathcal{A} = \partial_x$ and $\mathcal{B} = \partial_x^3 + 2\bar{u}_x\partial_x + \bar{u}_{xx}$. Let $t_0 = x$

$$\begin{tikzcd}
\bar{u}_{t_0} = \bar{u}_x \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & \bar{u}_{xx} = -\frac{\delta h_0}{\delta \bar{u}} & h_0 = -\frac{1}{2}\bar{u}_x^2 \\
\bar{u}_{t_1} = \frac{3}{2}\bar{u}_x^2 + \bar{u}_{xxx} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & 3\bar{u}_x\bar{u}_{xx} + \bar{u}_{xxxx} = -\frac{\delta h_1}{\delta \bar{u}} & h_1 = -\frac{1}{2}\bar{u}_x^3 + \frac{1}{2}\bar{u}_{xx}^2 \\
\bar{u}_{t_2} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_2}{\delta \bar{u}} & h_2 = -\frac{5}{8}\bar{u}_x^4 + \frac{5}{2}\bar{u}_x\bar{u}_{xx}^2 - \frac{1}{2}\bar{u}_{xxx}^2 \\
\dots & \dots &
\end{tikzcd}$$

$$\bar{u}_{t_2} = \frac{5}{2}\bar{u}_x^3 + \frac{5}{2}\bar{u}_{xx}^2 + 5\bar{u}_x\bar{u}_{xxx} + \bar{u}_{xxxxx},$$

$$\bar{u}_{t_3} = \frac{35}{8}\bar{u}_x^4 + \frac{35}{2}\bar{u}_x\bar{u}_{xx}^2 + \frac{35}{2}\bar{u}_x^2\bar{u}_{xxx} + \frac{21}{2}\bar{u}_{xxx}^2 + 14\bar{u}_{xx}\bar{u}_{xxxx} + 7\bar{u}_x\bar{u}_{5x} + \bar{u}_{7x},$$

$$h_3 = -\frac{7}{8}\bar{u}_x^5 + \frac{35}{4}\bar{u}_x^2\bar{u}_{xx}^2 - \frac{7}{2}\bar{u}_x\bar{u}_{xxx}^2 + \frac{1}{2}\bar{u}_{xxxx}^2.$$

First multiform

$$L_{0j} = \tfrac{1}{2}\,\bar{u}_x\,\bar{u}_{t_j} - h_j[\bar{u}]$$

$$\begin{aligned}
L_{01} &= \tfrac{1}{2}\,\bar{u}_x\,\bar{u}_{t_1} - \tfrac{1}{2}\,\bar{u}_x^3 + \tfrac{1}{2}\,\bar{u}_{xx}^2\\
L_{02} &= \tfrac{1}{2}\,\bar{u}_x\,\bar{u}_{t_2} - \tfrac{5}{8}\,\bar{u}_x^4 + \tfrac{5}{2}\,\bar{u}_x\,\bar{u}_{xx}^2 - \tfrac{1}{2}\,\bar{u}_{xxx}^2\\
L_{03} &= \tfrac{1}{2}\,\bar{u}_x\,\bar{u}_{t_3} - \tfrac{7}{8}\,\bar{u}_x^5 + \tfrac{35}{4}\,\bar{u}_x^2\,\bar{u}_{xx}^2 - \tfrac{7}{2}\,\bar{u}_x\,\bar{u}_{xxx}^2 + \tfrac{1}{2}\,\bar{u}_{xxxx}^2
\end{aligned}$$

By construction, $\frac{\delta_{0j} L_{0j}}{\delta \bar{u}} = -\mathcal{A}(\bar{u}_{t_j} - Q_j)$.

L_{ij} for $i, j > 0$ are not pretty but easily computed, for example

$$\begin{aligned}
L_{12} = {} & \tfrac{3}{8}\,\bar{u}_x^5 - \tfrac{15}{8}\,\bar{u}_x^2\,\bar{u}_{xx}^2 + \tfrac{5}{2}\,\bar{u}_x^3\,\bar{u}_{xxx} - \tfrac{5}{4}\,\bar{u}_x^3\,\bar{u}_{t_1} + \tfrac{7}{4}\,\bar{u}_{xx}^2\,\bar{u}_{xxx} + \tfrac{3}{2}\,\bar{u}_x\,\bar{u}_{xxx}^2 - 3\bar{u}_x\,\bar{u}_{xx}\,\bar{u}_{xxxx}\\
& + \tfrac{3}{4}\,\bar{u}_x^2\,\bar{u}_{xxxxx} + 5\bar{u}_x\,\bar{u}_{xx}\,\bar{u}_{xt_1} - \tfrac{5}{4}\,\bar{u}_{xx}^2\,\bar{u}_{t_1} - \tfrac{5}{2}\,\bar{u}_x\,\bar{u}_{xxx}\,\bar{u}_{t_1} + \tfrac{3}{4}\,\bar{u}_x^2\,\bar{u}_{t_2} - \tfrac{1}{2}\,\bar{u}_{xxxx}^2\\
& + \tfrac{1}{2}\,\bar{u}_{xxx}\,\bar{u}_{xxxxx} - \bar{u}_{xxx}\,\bar{u}_{xxt_1} + \bar{u}_{xxxx}\,\bar{u}_{xt_1} - \bar{u}_{xx}\,\bar{u}_{xt_2} - \tfrac{1}{2}\,\bar{u}_{xxxxx}\,\bar{u}_{t_1} + \tfrac{1}{2}\,\bar{u}_{xxx}\,\bar{u}_{t_2}
\end{aligned}$$

so

$$\frac{\delta_{01} L_{01}}{\delta \bar{u}_x} + \frac{\delta_{12} L_{12}}{\delta \bar{u}_{t_2}} = \left(\tfrac{1}{2}\,\bar{u}_{t_1} - \tfrac{3}{2}\,\bar{u}_x^2 - \bar{u}_{xxx}\right) + \left(\tfrac{3}{4}\,\bar{u}_x^2 + \tfrac{1}{2}\,\bar{u}_{xxx}\right) = \tfrac{1}{2}\left(\bar{u}_{t_1} - \tfrac{3}{2}\,\bar{u}_x^2 - \bar{u}_{xxx}\right)$$

And, if you really want to know...

$$\begin{aligned}
L_{13} = {} & \tfrac{35}{32}\,\bar{u}_{x}^{6} - \tfrac{35}{8}\,\bar{u}_{x}^{3}\,\bar{u}_{xx}^{2} + \tfrac{175}{16}\,\bar{u}_{x}^{4}\,\bar{u}_{xxx} - \tfrac{35}{16}\,\bar{u}_{x}^{4}\,\bar{u}_{t_{\mathbf{1}}} + \tfrac{35}{8}\,\bar{u}_{xx}^{4} - \tfrac{35}{4}\,\bar{u}_{x}\,\bar{u}_{xx}^{2}\,\bar{u}_{xxx} + \tfrac{147}{8}\,\bar{u}_{x}^{2}\,\bar{u}_{xxx}^{2} \\
& - \tfrac{21}{2}\,\bar{u}_{x}^{2}\,\bar{u}_{xx}\,\bar{u}_{xxxx} + \tfrac{21}{4}\,\bar{u}_{x}^{3}\,\bar{u}_{xxxxx} + \tfrac{35}{2}\,\bar{u}_{x}^{2}\,\bar{u}_{xx}\,\bar{u}_{xt_{\mathbf{1}}} - \tfrac{35}{4}\,\bar{u}_{x}\,\bar{u}_{xx}^{2}\,\bar{u}_{t_{\mathbf{1}}} - \tfrac{35}{4}\,\bar{u}_{x}^{2}\,\bar{u}_{xxx}\,\bar{u}_{t_{\mathbf{1}}} \\
& + \tfrac{19}{4}\,\bar{u}_{xxx}^{3} - 2\bar{u}_{xx}\,\bar{u}_{xxx}\,\bar{u}_{xxxx} - 5\bar{u}_{x}\,\bar{u}_{xxxx}^{2} + 3\bar{u}_{xx}^{2}\,\bar{u}_{xxxxx} + \tfrac{13}{2}\,\bar{u}_{x}\,\bar{u}_{xxx}\,\bar{u}_{xxxxx} - 3\bar{u}_{x}\,\bar{u}_{xx}\,\bar{u}_{xxxxxx} \\
& + \tfrac{3}{4}\,\bar{u}_{x}^{2}\,\bar{u}_{xxxxxxx} - 7\bar{u}_{x}\,\bar{u}_{xxx}\,\bar{u}_{xxt_{\mathbf{1}}} + 7\bar{u}_{xx}\,\bar{u}_{xxx}\,\bar{u}_{xt_{\mathbf{1}}} + 7\bar{u}_{x}\,\bar{u}_{xxxx}\,\bar{u}_{xt_{\mathbf{1}}} - \tfrac{21}{4}\,\bar{u}_{xxx}^{2}\,\bar{u}_{t_{\mathbf{1}}} - 7\bar{u}_{xx}\,\bar{u}_{xxxx}\,\bar{u}_{t_{\mathbf{1}}} \\
& - \tfrac{7}{2}\,\bar{u}_{x}\,\bar{u}_{xxxxx}\,\bar{u}_{t_{\mathbf{1}}} + \tfrac{3}{4}\,\bar{u}_{x}^{2}\,\bar{u}_{t_{\mathbf{3}}} + \tfrac{1}{2}\,\bar{u}_{xxxxx}^{2} - \bar{u}_{xxxx}\,\bar{u}_{xxxxxx} + \tfrac{1}{2}\,\bar{u}_{xxx}\,\bar{u}_{xxxxxxx} + \bar{u}_{xxxx}\,\bar{u}_{xxxt_{\mathbf{1}}} \\
& - \bar{u}_{xxxxx}\,\bar{u}_{xxt_{\mathbf{1}}} + \bar{u}_{xxxxxx}\,\bar{u}_{xt_{\mathbf{1}}} - \bar{u}_{xx}\,\bar{u}_{xt_{\mathbf{3}}} - \tfrac{1}{2}\,\bar{u}_{xxxxxxx}\,\bar{u}_{t_{\mathbf{1}}} + \tfrac{1}{2}\,\bar{u}_{xxx}\,\bar{u}_{t_{\mathbf{3}}}
\end{aligned}$$

$$\begin{aligned}
L_{23} = {} & \tfrac{25}{32}\,\bar{u}_{x}^{7} - \tfrac{175}{32}\,\bar{u}_{x}^{4}\,\bar{u}_{xx}^{2} + \tfrac{175}{16}\,\bar{u}_{x}^{5}\,\bar{u}_{xxx} + \tfrac{175}{8}\,\bar{u}_{x}\,\bar{u}_{xx}^{4} - \tfrac{175}{8}\,\bar{u}_{x}^{2}\,\bar{u}_{xx}^{2}\,\bar{u}_{xxx} + \tfrac{245}{8}\,\bar{u}_{x}^{3}\,\bar{u}_{xxx}^{2} \\
& - \tfrac{35}{2}\,\bar{u}_{x}^{3}\,\bar{u}_{xx}\,\bar{u}_{xxxx} + \tfrac{105}{16}\,\bar{u}_{x}^{4}\,\bar{u}_{xxxxx} - \tfrac{35}{16}\,\bar{u}_{x}^{4}\,\bar{u}_{t_{\mathbf{2}}} - \tfrac{305}{8}\,\bar{u}_{xx}^{2}\,\bar{u}_{xxx}^{2} + \tfrac{95}{4}\,\bar{u}_{x}\,\bar{u}_{xxx}^{3} + 20\bar{u}_{xx}^{3}\,\bar{u}_{xxxx} \\
& - 10\bar{u}_{x}\,\bar{u}_{xx}\,\bar{u}_{xxx}\,\bar{u}_{xxxx} - \tfrac{25}{2}\,\bar{u}_{x}^{2}\,\bar{u}_{xxxx}^{2} + 15\bar{u}_{x}\,\bar{u}_{xx}^{2}\,\bar{u}_{xxxxx} + \tfrac{65}{4}\,\bar{u}_{x}^{2}\,\bar{u}_{xxx}\,\bar{u}_{xxxxx} - \tfrac{15}{2}\,\bar{u}_{x}^{2}\,\bar{u}_{xx}\,\bar{u}_{xxxxxx} \\
& + \tfrac{5}{4}\,\bar{u}_{x}^{3}\,\bar{u}_{xxxxxxx} + \tfrac{35}{2}\,\bar{u}_{x}^{2}\,\bar{u}_{xx}\,\bar{u}_{xt_{\mathbf{2}}} - \tfrac{35}{4}\,\bar{u}_{x}\,\bar{u}_{xx}^{2}\,\bar{u}_{t_{\mathbf{2}}} - \tfrac{35}{4}\,\bar{u}_{x}^{2}\,\bar{u}_{xxx}\,\bar{u}_{t_{\mathbf{2}}} + \tfrac{5}{4}\,\bar{u}_{x}^{3}\,\bar{u}_{t_{\mathbf{3}}} + \tfrac{19}{4}\,\bar{u}_{xxx}^{2}\,\bar{u}_{xxxxx} \\
& + 8\bar{u}_{xx}\,\bar{u}_{xxxx}\,\bar{u}_{xxxxx} + \tfrac{5}{2}\,\bar{u}_{x}\,\bar{u}_{xxxxx}^{2} - 10\bar{u}_{xx}\,\bar{u}_{xxx}\,\bar{u}_{xxxxxx} - 5\bar{u}_{x}\,\bar{u}_{xxxx}\,\bar{u}_{xxxxxx} + \tfrac{5}{4}\,\bar{u}_{xx}^{2}\,\bar{u}_{xxxxxxx} \\
& + \tfrac{5}{2}\,\bar{u}_{x}\,\bar{u}_{xxx}\,\bar{u}_{xxxxxxx} - 7\bar{u}_{x}\,\bar{u}_{xxx}\,\bar{u}_{xxt_{\mathbf{2}}} + 7\bar{u}_{xx}\,\bar{u}_{xxx}\,\bar{u}_{xt_{\mathbf{2}}} + 7\bar{u}_{x}\,\bar{u}_{xxxx}\,\bar{u}_{xt_{\mathbf{2}}} - 5\bar{u}_{x}\,\bar{u}_{xx}\,\bar{u}_{xt_{\mathbf{3}}} - \tfrac{21}{4}\,\bar{u}_{xxx}^{2}\,\bar{u}_{t_{\mathbf{2}}} \\
& - 7\bar{u}_{xx}\,\bar{u}_{xxxx}\,\bar{u}_{t_{\mathbf{2}}} - \tfrac{7}{2}\,\bar{u}_{x}\,\bar{u}_{xxxxx}\,\bar{u}_{t_{\mathbf{2}}} + \tfrac{5}{4}\,\bar{u}_{xx}^{2}\,\bar{u}_{t_{\mathbf{3}}} + \tfrac{5}{2}\,\bar{u}_{x}\,\bar{u}_{xxx}\,\bar{u}_{t_{\mathbf{3}}} - \tfrac{1}{2}\,\bar{u}_{xxxxxx}^{2} + \tfrac{1}{2}\,\bar{u}_{xxxxx}\,\bar{u}_{xxxxxxx} \\
& + \bar{u}_{xxxx}\,\bar{u}_{xxxt_{\mathbf{2}}} - \bar{u}_{xxxxx}\,\bar{u}_{xxt_{\mathbf{2}}} + \bar{u}_{xxx}\,\bar{u}_{xxt_{\mathbf{3}}} + \bar{u}_{xxxxxx}\,\bar{u}_{xt_{\mathbf{2}}} - \bar{u}_{xxxx}\,\bar{u}_{xt_{\mathbf{3}}} - \tfrac{1}{2}\,\bar{u}_{xxxxxxx}\,\bar{u}_{t_{\mathbf{2}}} + \tfrac{1}{2}\,\bar{u}_{xxxxx}\,\bar{u}_{t_{\mathbf{3}}}
\end{aligned}$$

Second multiform

$L_{0j} = \frac{1}{2}\left(\bar{u}_x^2 + \bar{u}_{xxx}\right)\bar{u}_{t_j} - h_{j+1}$:

$$L_{01} = \frac{1}{2}\left(\bar{u}_x^2 + \bar{u}_{xxx}\right)\bar{u}_{t_1} - \frac{5}{8}\bar{u}_x^4 + \frac{5}{2}\bar{u}_x\bar{u}_{xx}^2 - \frac{1}{2}\bar{u}_{xxx}^2$$

$$L_{02} = \frac{1}{2}\left(\bar{u}_x^2 + \bar{u}_{xxx}\right)\bar{u}_{t_2} - \frac{7}{8}\bar{u}_x^5 + \frac{35}{4}\bar{u}_x^2\bar{u}_{xx}^2 - \frac{7}{2}\bar{u}_x\bar{u}_{xxx}^2 + \frac{1}{2}\bar{u}_{xxxx}^2$$

By construction, $\frac{\delta_{0j}L_{0j}}{\delta\bar{u}} = -\mathcal{B}(\bar{u}_{t_j} - Q_j)$.

L_{ij} for $i,j>0$ are horrendous but easily computed, for example

$$\begin{aligned}
L_{12} = {} & \frac{5}{8}\bar{u}_x^6 + \frac{5}{4}\bar{u}_x^3\bar{u}_{xx}^2 + \frac{55}{8}\bar{u}_x^4\bar{u}_{xxx} - \frac{15}{8}\bar{u}_x^4\bar{u}_{t_1} + \frac{5}{8}\bar{u}_{xx}^4 - \frac{25}{4}\bar{u}_x\bar{u}_{xx}^2\bar{u}_{xxx} + \frac{45}{4}\bar{u}_x^2\bar{u}_{xxx}^2 \\
& + 4\bar{u}_x^3\bar{u}_{xxxxx} + \frac{5}{4}\bar{u}_x^3\bar{u}_{xxt_1} + \frac{55}{4}\bar{u}_x^2\bar{u}_{xx}\bar{u}_{xt_1} - \frac{15}{2}\bar{u}_x\bar{u}_{xx}^2\bar{u}_{t_1} - \frac{35}{4}\bar{u}_x^2\bar{u}_{xxx}\bar{u}_{t_1} + \bar{u}_x^3\bar{u}_{t_2} \\
& + 5\bar{u}_{xxx}^3 + \frac{5}{2}\bar{u}_{xx}\bar{u}_{xxx}\bar{u}_{xxxx} - \frac{5}{2}\bar{u}_x\bar{u}_{xxxx}^2 + \frac{1}{4}\bar{u}_{xx}^2\bar{u}_{xxxxx} + \frac{5}{2}\bar{u}_x\bar{u}_{xxx}\bar{u}_{xxxxx} - \frac{3}{2}\bar{u}_x\bar{u}_{xx}\bar{u}_{xxxxxx} \\
& + \frac{3}{4}\bar{u}_x^2\bar{u}_{xxxxxxx} + \frac{5}{4}\bar{u}_{xx}^2\bar{u}_{xxt_1} - \frac{9}{2}\bar{u}_x\bar{u}_{xxx}\bar{u}_{xxt_1} - \frac{3}{4}\bar{u}_x^2\bar{u}_{xxt_2} + 2\bar{u}_{xx}\bar{u}_{xxx}\bar{u}_{xt_1} + \frac{9}{2}\bar{u}_x\bar{u}_{xxxx}\bar{u}_{xt_1} \\
& - \frac{7}{2}\bar{u}_x\bar{u}_{xx}\bar{u}_{xt_2} - \frac{11}{2}\bar{u}_{xxx}^2\bar{u}_{t_1} - \frac{13}{2}\bar{u}_{xx}\bar{u}_{xxxx}\bar{u}_{t_1} - \frac{7}{2}\bar{u}_x\bar{u}_{xxxxx}\bar{u}_{t_1} + \bar{u}_{xx}^2\bar{u}_{t_2} + \frac{5}{2}\bar{u}_x\bar{u}_{xxx}\bar{u}_{t_2} \\
& - \frac{1}{2}\bar{u}_{xxxx}\bar{u}_{xxxxxx} + \frac{1}{2}\bar{u}_{xxx}\bar{u}_{xxxxxxx} + \bar{u}_{xxxx}\bar{u}_{xxxt_1} - \frac{1}{2}\bar{u}_{xxxxx}\bar{u}_{xxt_1} \\
& + \frac{1}{2}\bar{u}_{xxx}\bar{u}_{xxt_2} + \frac{1}{2}\bar{u}_{xxxxxx}\bar{u}_{xt_1} - \frac{1}{2}\bar{u}_{xxxx}\bar{u}_{xt_2} - \frac{1}{2}\bar{u}_{xxxxxxx}\bar{u}_{t_1} + \frac{1}{2}\bar{u}_{xxxxx}\bar{u}_{t_2}
\end{aligned}$$

so

$$\frac{\delta_{01}L_{01}}{\delta\bar{u}_{xxx}} + \frac{\delta_{12}L_{12}}{\delta\bar{u}_{xxt_2}} = \left(\frac{1}{2}\bar{u}_{t_1} - \bar{u}_{xxx}\right) + \left(-\frac{3}{4}\bar{u}_x^2 + \frac{1}{2}\bar{u}_{xxx}\right) = \frac{1}{2}\left(\bar{u}_{t_1} - \frac{3}{2}\bar{u}_x^2 - \bar{u}_{xxx}\right)$$

Contents

Other examples

Nutku & Pavlov. Multi-Lagrangians for integrable systems. 2002.

Present bi-Lagrangians for

- KdV
- polytropic gas dynamics
- Kaup-Boussinesq
- NLS
- . . .

These are the same Lagrangians as we find, but their presentation suggest some educated guesswork was involved.

We seem to have a general method.

Multi-Lagrangian $\neq$ Lagrangian multiform

Can we forget about the Hamiltonian side?

How to express the compatiblity of two Lagrangians / two multiforms / two symplectic operators?

Invertible symplectic operators $\mathcal{I}$ and $\mathcal{J}$ are compatible

- if $\mathcal{I}^{-1}$ and $\mathcal{J}^{-1}$ are compatible Hamiltonian operators
- if $\mathcal{I}^{-1} + \lambda \mathcal{J}^{-1}$ is Hamiltonian
- if
$$(\mathcal{I}^{-1} + \lambda \mathcal{J}^{-1})^{-1} = \mathcal{I} + \lambda \mathcal{I}\mathcal{J}^{-1}\mathcal{I} + \lambda^2 \mathcal{I}\mathcal{J}^{-1}\mathcal{I}\mathcal{J}^{-1}\mathcal{I} + \dots$$
is symplectic
- if $\mathcal{I}, \quad \mathcal{I}\mathcal{J}^{-1}\mathcal{I}, \quad \mathcal{I}\mathcal{J}^{-1}\mathcal{I}\mathcal{J}^{-1}\mathcal{I}, \dots$ are symplectic
- if $\mathcal{I}\mathcal{J}^{-1}$ is a Nijenhuis operator

Conclusions

- A “potential” variable should be defined by a Hamiltonian operator.

  It is in a space dual to the original variable.
- This allows us to transform Hamiltonian operators into symplectic operators and find Lagrangians
- In the traditional calculus of variations, the “higher Lagrangians” give increasingly weak differential consequences.

  In Lagrangian multiform theory, all give evolutionary equations.

References

- Dorfman. Dirac structures of integrable evolution equations. 1987.
- Mokhov. Symplectic and Poisson structures on loop spaces of smooth manifolds, and integrable systems. 1998.
- Nutku & Pavlov. Multi-Lagrangians for integrable systems. 2002.
- Suris & V. On the Lagrangian structure of integrable hierarchies. 2016
- Vergallo & V. In preparation.

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Thank you for your attention!