An invitation to Lagrangian multiforms

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Integrable systems: regularity, non-commutativity and random matrix theory

ICMS

18 June 2026

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2. Lagrangian 1-forms and integrable ODEs
3. Lagrangian multiforms for integrable PDEs
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5. Non-commutativity
6. Recent progress and open questions

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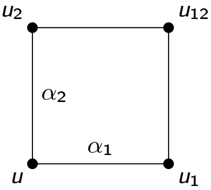
2003: Quad equations and consistency around the cube

$$Q(u, u_1, u_2, u_{12}, \alpha_1, \alpha_2) = 0$$

Subscripts of u denote lattice shifts, α_i are parameters.

Invariant under symmetries of the square

Affine in each of u, u_1, u_2, u_{12}.



Example (lattice potential KdV):

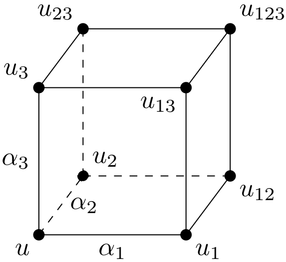
$$(u - u_{12})(u_1 - u_2) - \alpha_1 + \alpha_2 = 0$$

Multi-dimensional consistency

The three ways of calculating u_{123}, using

$$Q(u, u_i, u_j, u_{ij}, \alpha_i, \alpha_j) = 0, \qquad i, j \in \{1, 2, 3\}$$

and its shifts, give the same result.



$\hookrightarrow$ Discrete analogue of commuting flows

Classification (under some extra assumptions) in the “ABS list”

[VE Adler, AI Bobenko, YB Suris. 2003]

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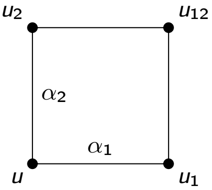

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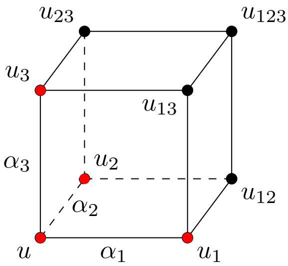

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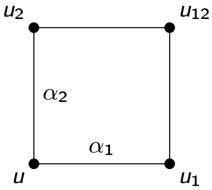

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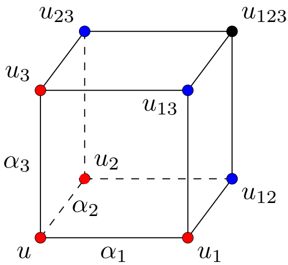

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2009: Lagrangian multiforms

Lagrangian $\mathcal{L}(u, u_i, u_j, u_{ij}, \alpha_i, \alpha_j)$ on each face

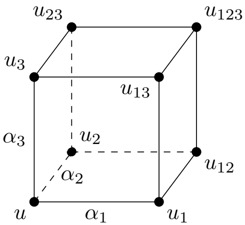
Action is sum over six faces of unit cube:

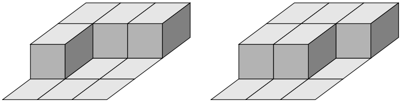
$$\begin{aligned} S = {} & \mathcal{L}(u, u_i, u_j, u_{ij}, \alpha_i, \alpha_j) + \mathcal{L}(u, u_j, u_k, u_{jk}, \alpha_j, \alpha_k) + \mathcal{L}(u, u_k, u_i, u_{ik}, \alpha_k, \alpha_i) \\ & - \mathcal{L}(u_k, u_{ik}, u_{jk}, u_{ijk}, \alpha_i, \alpha_j) - \; cL(u_i, u_{ij}, u_{ik}, u_{ijk}, \alpha_j, \alpha_k) - \mathcal{L}(u_j, u_{jk}, u_{ij}, u_{ijk}, \alpha_k, \alpha_i) \end{aligned}$$

- Pluri-Lagrangian principle (corner equations):

  $$\frac{\partial S}{\partial v} = 0 \qquad \text{for } v \in \{u, u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}, u_{ijk}\}$$

- Lagrangian multiform principle:
  additionally require closure: $S = 0$





If action critical on all unit cubes, then it is critical on every closed discrete surface.

[S Lobb, F Nijhoff. 2009]

That escalated... Incomplete list – more to come in final part

Discrete 2-forms:

[S Lobb, F Nijhoff. 2009, 2010, 2018], [AI Bobenko, YB Suris. 2010, 2015]
[P Xenitidis, F Nijhoff, S Lobb. 2011], [J Atkinson, SB Lobb, FW Nijhoff. 2012]
[R Boll, M Petrera, YB Suris. 2014, 2016], [SD King, FW Nijhoff. 2019]
[JJ Richardson, MV. 2025]

Continuous 2-forms:

[YB Suris, MV. 2016], [MV, 2019, 2021], [M Petrera, MV. 2020]
[D Sleigh, F Nijhoff, V Caudrelier.2019, 2020], [V Caudrelier, M Stoppato. 2020, 2021]

Discrete and continuous 1-forms:

[S Yoo-Kong, S Lobb, F Nijhoff. 2011], [YB Suris. 2013, 2016]
[R Boll, M Petrera, YB Suris, 2013, 2015], [M Petrera, YB Suris. 2017]
[U Jairuk, M Tanasittikosol, S Yoo-Kong. 2017]

Discrete and continuous 3-forms:

[SB Lobb, FW Nijhoff, GRW Quispel. 2009], [R Boll, M Petrera, YB Suris. 2016]
[D Sleigh, FW Nijhoff, V Caudrelier. 2023], [FW Nijhoff. 2023]

Contents

Liouville integrability

A Hamiltonian system with Hamilton function $H : T^*Q \cong \mathbb{R}^{2N} \to \mathbb{R}$ is Liouville integrable if there exist N functionally independent Hamilton functions $H = H_1, H_2, \dots H_N$ such that $\{H_i, H_j\} = 0$.

- Each H_i defines its own flow $\phi^t_{H_j}$: N dynamical systems
- Each H_i is a conserved quantity for all flows
- Each common level set (if compact and nondegenerate) is a torus
- The flows commute

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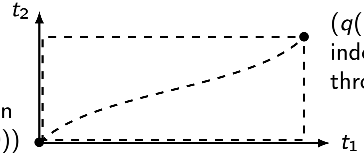
We can consider (q, p) as a function of multi-time, $\mathbb{R}^N \to T^*Q$:

$$(t_1, \dots, t_N) \mapsto (q(t_1, \dots, t_N), p(t_1, \dots, t_N))$$



$(q(t_1, t_2), p(t_1, t_2))$
independent of path
through multi-time

Initial condition
$(q(0, 0), p(0, 0))$

Variational principle for commuting flows

Suppose we have Lagrange functions L_i associated to H_i. Consider

$$q : \mathbb{R}^N \to Q \quad \text{(multi-time to configuration space)}$$

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Pluri-Lagrangian principle

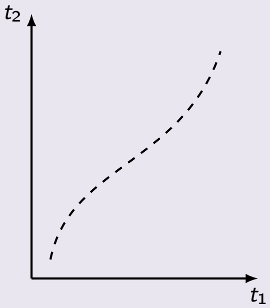
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q] \, \mathrm{d}t_i .$$

Look for $q(t_1, \ldots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$



Lagrangian multiform principle

Considers variations of the curve too: $\mathrm{d}\mathcal{L} = 0$ on solutions.

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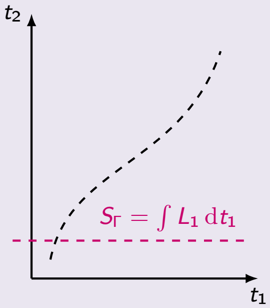

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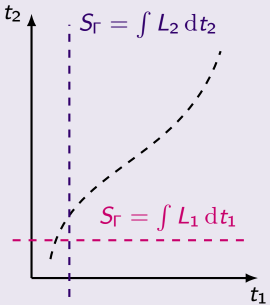

Lagrangian multiform principle

Considers variations of the curve too: $\mathrm{d}\mathcal{L} = 0$ on solutions.

Multiform Euler-Lagrange equations

Assume that

$$\begin{aligned} &L_1[q] = L_1(q, q_{t_1}), \\ &L_i[q] = L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1 \end{aligned}$$

and consider the 1-form

$$\mathcal{L} = \sum_i L_i[q] \, \mathrm{d}t_i$$

The equations characterising the pluri-Lagrangian principle are:

Multiform Euler-Lagrange equations

Usual Euler-Lagrange equations: $\dfrac{\partial L_i}{\partial q} - \dfrac{\mathrm{d}}{\mathrm{d}t_i} \dfrac{\partial L_i}{\partial q_{t_i}} = 0$

EL equations wrt q_{t_1} instead of q : $\dfrac{\partial L_i}{\partial q_{t_1}} = 0, \qquad i \neq 1$

Compatibility conditions: $\dfrac{\partial L_i}{\partial q_{t_i}} = \dfrac{\partial L_j}{\partial q_{t_j}}$

(No summation over repeated indices)

Example: Kepler Problem

Take

$$\begin{aligned} L_1 &= \frac{1}{2}|q_{t_1}|^2 + \frac{1}{|q|} \\ L_2 &= q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot \hat{v} \qquad (\hat{v} \text{ fixed unit vector}) \end{aligned}$$

In general, for systems of Newtonian type, $L_i = q_{t_1} q_{t_i} - H_i(q, q_{t_1})$

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In general, for systems of Newtonian type, $L_i = q_{t_1} q_{t_i} - H_i(q, q_{t_1})$

Multiform Euler-Lagrange equations of $\mathcal{L} = L_1 \mathrm{d}t_1 + L_2 \mathrm{d}t_2$

$$\frac{\partial L_1}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_1}\frac{\partial L_1}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_1 t_1} = -\frac{q}{|q|^3} \qquad \text{(Keplerian motion)}$$

$$\frac{\partial L_2}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_2}\frac{\partial L_2}{\partial q_{t_2}} = 0 \quad \Rightarrow \quad q_{t_1 t_2} = \hat{v} \times q_{t_1}$$

$$\frac{\partial L_2}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_2} = \hat{v} \times q \qquad \text{(Rotation)}$$

$$\frac{\partial L_1}{\partial q_{t_1}} = \frac{\partial L_2}{\partial q_{t_2}} \quad \Rightarrow \quad q_{t_1} = q_{t_1}$$

Multiform Euler-Lagrange equations

When $L_1[q] = L_1(q, q_{t_1})$ and $L_i[q] = L_i(q, q_{t_1}, q_{t_i})$:

Usual Euler-Lagrange equations: $$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial L_i}{\partial q_{t_i}} = 0$$

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When L_i depend on arbitrary derivatives of q

$$\frac{\delta_i L_i}{\delta q_I} = 0 \qquad \forall I \not\ni t_i$$

$$\frac{\delta_i L_i}{\delta q_{It_i}} = \frac{\delta_j L_j}{\delta q_{It_j}} \qquad \forall I$$

where where I denotes a multi-index (a combination of derivatives) and

$$\frac{\delta_i}{\delta q_I} = \sum_{\alpha \geq 0} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\partial}{\partial q_{It_i^\alpha}} = \frac{\partial}{\partial q_I} - \frac{\mathrm{d}}{\mathrm{d}t_i}\frac{\partial}{\partial q_{It_i}} + \dots$$

Contents

Variational principle for 2-forms

Pluri-Lagrangian principle

Given a 2-form

$$\mathcal{L}[u] = \textstyle\sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

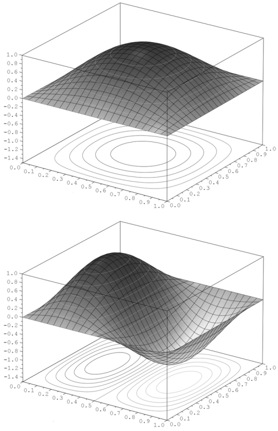
find a field $u(t_1, \ldots t_N)$, such that

$$\int_\Gamma \mathcal{L}[u]$$

is critical on all smooth surfaces Γ in multi-time $\mathbb{R}^N$, w.r.t. variations of u.

Lagrangian multiform principle

Additionally require that $\mathrm{d}\mathcal{L}[u] = 0$ on solutions: action is critical w.r.t. deformations of the surface.



Multiform EL equations for a 2-form

Let $\mathcal{L}[u] = \sum_{i,j} L_{ij}[u] \,\mathrm{d}t_i \wedge \mathrm{d}t_j$.

Solutions to the pluri-Lagrangian principle are characterised by

$$\begin{aligned}
&\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 && \forall I \not\ni t_i, t_j, \\
&\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} && \forall I \not\ni t_i, \\
&\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 && \forall I,
\end{aligned}$$

where I denotes a multi-index (a combination of derivatives) and

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial u_{It_i^\alpha t_j^\beta}}$$

Exterior derivative

If the surface of integration is the boundary of a volume, $\Gamma = \partial B$, then

$$\int_\Gamma \mathcal{L} = \int_B \mathrm{d}\mathcal{L}.$$

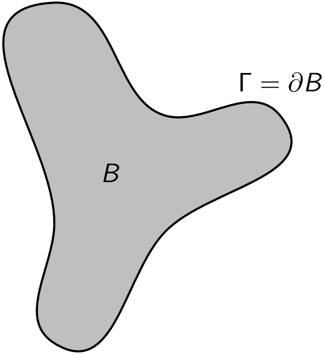
So a necessary (and in fact sufficient) condition for criticality is that infinitesimal variations of $\mathrm{d}\mathcal{L}$ vanish:

$$\delta \mathrm{d}\mathcal{L} = 0.$$

In other words:

$$\frac{\partial P_{ijk}}{\partial u_I} = 0 \qquad \forall I, \qquad\qquad (*)$$

where $\mathrm{d}\mathcal{L} = \sum_{i,j,k} P_{ijk}\, \mathrm{d}t^i \wedge \mathrm{d}t^i \wedge \mathrm{d}t^k$.



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Double-zero property

If the coefficients of $\mathrm{d}\mathcal{L}$ factorise,

$$\mathrm{d}\mathcal{L} = \sum_{i<j<k} A_{ijk} B_{ijk}\, \mathrm{d}t^i \wedge \mathrm{d}t^j \wedge \mathrm{d}t^k,$$

then on solutions of the system

$$A_{ijk} = 0, \qquad B_{ijk} = 0,$$

condition $(*)$ is satisfied:

$$\frac{\partial P_{ijk}}{\partial u_I} = \frac{\partial A_{ijk}}{\partial u_I} B_{ijk} + A_{ijk} \frac{\partial B_{ijk}}{\partial u_I} = 0$$

Hence the system $A_{ijk} = 0$, $B_{ijk} = 0$ implies the multiform Euler-Lagrange equations

Contents

Variational symmetries → Lagrangian multiforms

Consider a Lagrangian L_{12} with variational symmetry $u_{t_3} = Q[u]$.

This means that $\int L_{12} \, \mathrm{d}t_1 \wedge \mathrm{d}t_2$ is invariant:

$$\int \underbrace{\mathrm{D}_{L_{12}}(Q)}_{\text{Fréchet derivative}} \, \mathrm{d}t_1 \wedge \mathrm{d}t_2 = \int \frac{\delta L_{12}}{\delta u} Q \, \mathrm{d}t_1 \wedge \mathrm{d}t_2 = 0 \qquad \Leftrightarrow \qquad \frac{\delta L_{12}}{\delta u} Q = \frac{\mathrm{d}M}{\mathrm{d}t_2} + \frac{\mathrm{d}N}{\mathrm{d}t_1}$$

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Combine this with

$$\int \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} \mathrm{d}t_1 \wedge \mathrm{d}t_2 = \int \frac{\delta L_{12}}{\delta u} u_{t_3} \, \mathrm{d}t_1 \wedge \mathrm{d}t_2 \qquad \Leftrightarrow \qquad \frac{\delta L_{12}}{\delta u} u_{t_3} = \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} + \frac{\mathrm{d}\hat{M}}{\mathrm{d}t_2} + \frac{\mathrm{d}\hat{N}}{\mathrm{d}t_1}$$

to get

$$\frac{\mathrm{d}L_{23}}{\mathrm{d}t_1} - \frac{\mathrm{d}L_{13}}{\mathrm{d}t_2} + \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} = \underbrace{\frac{\delta L_{12}}{\delta u}}_{A} \underbrace{(u_{t_3} - Q)}_{B}$$

Double-zero property:

$$A = 0, \quad B = 0$$

implies the multiform Euler-Lagrange equations for

$$\mathcal{L} = L_{12} \, \mathrm{d}t_1 \wedge \mathrm{d}t_2 + L_{13} \, \mathrm{d}t_1 \wedge \mathrm{d}t_3 + L_{23} \, \mathrm{d}t_2 \wedge \mathrm{d}t_3$$

Example: potential KdV hierarchy

The potential KdV equation $u_2 = u_{111} + 3u_1^2$ has a (weak) Lagrangian

$$L_{12} = \tfrac{1}{2} u_1 u_2 - \tfrac{1}{2} u_1 u_{111} - u_1^3 \qquad (u_k = u_{t_k})$$

and a hierarchy of variational symmetries, starting with

$$u_3 = Q[u] := u_{11111} + 10 u_1 u_{111} + 5 u_{11}^2 + 10 u_1^3 .$$

This yields L_{13} and L_{23} such that

$$\frac{\mathrm{d} L_{23}}{\mathrm{d} t_1} - \frac{\mathrm{d} L_{13}}{\mathrm{d} t_2} + \frac{\mathrm{d} L_{12}}{\mathrm{d} t_3} = \frac{\delta L_{12}}{\delta u} (u_3 - Q)$$

Explicitly:

$$\begin{aligned}
L_{12} &= \tfrac{1}{2} u_1 u_2 - \tfrac{1}{2} u_1 u_{111} - u_1^3 \\
L_{13} &= \tfrac{1}{2} u_1 u_3 - \tfrac{5}{2} u_1^4 - 5 u_1 u_{11}^2 - 5 u_1^2 u_{111} - \tfrac{1}{2} u_{111}^2 \\
L_{23} &= -12 u_1^5 - 15 u_1^2 u_{11}^2 - 10 u_1^3 u_{111} + u_{11}^2 u_{111} - 2 u_1 u_{111}^2 - u_1 u_{11} u_{1111} - 5 u_1^2 u_{112} \\
&\quad + 3 u_1^2 u_3 - \tfrac{1}{2} u_{1111}^2 - u_{111} u_{112} + \tfrac{1}{2} u_1 u_{113} + u_{1111} u_{12} - \tfrac{1}{2} u_{11} u_{13} + u_{111} u_3 - \tfrac{1}{2} u_2 u_3
\end{aligned}$$

Multiform EL equations equivalent to $u_{12} = \frac{\mathrm{d}}{\mathrm{d}t_1} \left(u_{111} + 3u_1^2 \right)$ and $u_3 = \ldots$.

Continuum limits

Miwa shifts

Skew embedding of the mesh $\mathbb{Z}^N$, with lattice parameters $\lambda_1, \ldots, \lambda_N$ into multi-time $\mathbb{R}^N$:

$$U = U(\boldsymbol{n}) = u(t_1, t_2, \ldots, t_N),$$

$$U_i = U(\boldsymbol{n} + \mathfrak{e}_i) = u\left(t_1 - 2\lambda_i, t_2 + 2\frac{\lambda_i^2}{2}, \ldots, t_N + 2(-1)^N \frac{\lambda_i^N}{N} \right)$$

Expansion in λ_i turns quad equation into hierarchy of PDEs.

Continuum limit of Lagrangians needs additional step:

- Expansion in two parameters λ_i, λ_j
- Euler-MacLaurin formula to turn action sum into integral
- Coefficients of double power series give L_{ij}

Construction from Lax pairs or R-matrices…

…bring us to non-commutative setting

Contents

Non-commuting variables

Matrix-valued fields

Zakharov-Mikhailov Lagrangian: ($\bar{U}$, $\bar{V}$ in Jordan form.)

$$L_{\xi\eta} = \operatorname{tr}\left(\varphi^{-1}\varphi_\eta \bar{U} - \psi^{-1}\psi_\xi \bar{V} - \frac{\psi \bar{V} \psi^{-1} \varphi \bar{U} \varphi^{-1}}{a-b} \right)$$

yields zero curvature equation for $U = \dfrac{\varphi \bar{U} \varphi^{-1}}{\lambda - a}$ and $V = \dfrac{\psi \bar{V} \psi^{-1}}{\lambda - b}$.

Multiform interpretation in [D Sleigh, F Nijhoff, V Caudrelier. 2019]

Relation to R-matrix structure in [V Caudrelier, M Stoppato. 2021]

Group-valued fields

Di-algebra construction gives a group G acting by co-adjoint action on the dual Lie algebra $\mathfrak{g}^*$ of a Lie group related to G.

$\mathcal{L}$ is defined as a function of $\varphi \in G$, but naturally expressed in terms of $L = \operatorname{Ad}^*_\varphi \Lambda$, where $\Lambda \in \mathfrak{g}^*$ is fixed:

$$\mathcal{L}[\varphi] = \sum_k \left(\left\langle L, \phi_k \cdot \phi^{-1} \right\rangle - H_k(L) \right) \mathrm{d}t_k$$

[V Caudrelier, M Dell'Atti, AA Sing. 2024]

Non-commuting variables

Can we work on free algebra?

Consider tr as equivalence class of algebra elements whose difference is a (sum of) commutator(s).

Example: potential KdV

$$
\begin{aligned}
L_{12} &= \qquad \frac{1}{2} u_1 u_2 - \frac{1}{2} u_1 u_{111} - u_1^3 \\
L_{13} &= \qquad \frac{1}{2} u_1 u_3 - \frac{1}{2} u_{111}^2 + 5 u_1 u_{11}^2 - \frac{5}{2} u_1^4
\end{aligned}
$$

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L_{13} &= \operatorname{tr}\left(\frac{1}{2}u_1u_3 - \frac{1}{2}u_{111}^2 + 5u_1u_{11}^2 - \frac{5}{2}u_1^4\right)
\end{aligned}
$$

L_{23} has terms such as $u_1u_{11}u_{1111}$ where the order matters even under the trace.

$\rightsquigarrow$ construct using double-zero approach.

$u_2 = u_{111} + 3u_1^2$

$u_{12} = u_{1111} + 3(u_1u_{11} + u_{11}u_1)$

$u_3 = u_{11111} + 5(u_1u_{111} + u_{111}u_1) + 5u_{11} + 10u_1^3$

Counterexample: potential modified KdV

Commutative L_{12}, L_{13} are unambiguous under trace in free algebra, but do not produce an integrable non-commutative version of potential mKdV.

Non-commuting flows

Commutativity encoded into framework by taking $\mathbb{R}^n$ as multi-time (i.e. space of independent variables)

We can also use a Lie group G as multi-time.

Let its Lie algebra $\mathfrak{g}$ be generated by ξ_i with structure relations $[\xi_i, \xi_j] = \sum_k C_{ij}^k \xi_k$.
Cannot write $\mathcal{L} = \sum_i L_i \,\mathrm{d}t_i$. Instead, $L_i = \xi_i \lrcorner \mathcal{L}$.

Multiform EL equations on first jet:

1-form

$$\frac{\delta_j L_j}{\delta q} = 0$$

$$\frac{\partial L_j}{\partial q_k} = 0 \quad k \neq j$$

$$\frac{\partial L_j}{\partial q_j} = \frac{\partial L_k}{\partial q_k}$$

2-form

$$\frac{\delta_{jk} L_{jk}}{\delta q} + \sum_\ell C_{jk}^\ell p_\ell = 0$$

$$\frac{\partial L_{jk}}{\partial q_\ell} = 0 \quad \ell \neq j, k$$

$$\frac{\partial L_{j\ell}}{\partial q_j} = \frac{\partial L_{k\ell}}{\partial q_k} := p_\ell$$

Non-commuting flows example: Kepler problem

$L_i = \xi_i \lrcorner \mathcal{L} = q_1 \cdot q_i - H_i$ where

- H_1 is the Kepler Hamiltonian (in (q, q_1)-variables)
- H_2, H_3, H_4 are the components of the angular momentum
- H_5, H_6, H_7 are the components of the Runge-Lenz vector

The Hamiltonians do not Poisson commute, and

$$\partial_j L_i - \partial_i L_j = \xi_j(\xi_i \lrcorner \mathcal{L}) - \xi_j(\xi_i \lrcorner \mathcal{L}) \neq 0,$$

but cancels (on solutions) against the additional term in the definition of d:

$$\xi_j(\xi_i \lrcorner \mathcal{L}) - \xi_j(\xi_i \lrcorner \mathcal{L}) - [\xi_i, \xi_j] \lrcorner \mathcal{L} = 0.$$

Lagrangian multiforms do not care whether

- vector fields are independent
- or if system is integrable, super-integrable, or non-integrable.

Limitation

What if the commutation relations are not linear? Lie Groupoid?

Contents

Variations of the multiform variational principle

- “Original”: $\delta \int_\Gamma \mathcal{L}[q] = 0$ for all curves/surfaces in $\mathbb{R}^n$

  E.g. for 1-form case ($\mathcal{L} = L_i \mathrm{d}t^i$):

  $$s \in [0,1] \xrightarrow{\Gamma} \boldsymbol{t}(s) \in \mathbb{R}^n \xrightarrow{\Sigma} (\boldsymbol{q}(\boldsymbol{t}(s))) \in M.$$

  Find $\Sigma : \mathbb{R}^n \to M$ such that $\int_0^1 (\Sigma \circ \Gamma)^* \mathcal{L}$ is critical wrt variations of Σ and Γ for all curves $\Gamma : [0,1] \to \mathbb{R}^n$.

  [Nijhoff & ..., 2009–...] emphasised variations wrt Γ.

  [Suris & ..., 2010–...] pluri-Lagrangian principle: variations only wrt Σ.

- Univariational principle (1-form case in phase space, $\mathcal{L} = p_i \mathrm{d}q^i - H_i \mathrm{d}t^i$):

  $$s \in [0,1] \xrightarrow{\Gamma} \boldsymbol{\tau}(s) \in \mathbb{R}^n \xrightarrow{\Sigma} (\boldsymbol{q}(\boldsymbol{\tau}(s)), \boldsymbol{t}(\boldsymbol{\tau}(s))) \in M \times \mathbb{R}^n.$$

  Find $\Sigma : \mathbb{R}^n \to M \times \mathbb{R}^n$ such that $\int_0^1 (\Sigma \circ \Gamma)^* \mathcal{L}$ is critical wrt variations of Σ for all curves $\Gamma : [0,1] \to \mathbb{R}^n$.

  [V Caudrelier, D Harland. 2025]

  [V Caudrelier, D Harland, AA Singh, B Vicedo. 2026]

Variations of the multiform variational principle

- Pointwise: require $\mathrm{d}\mathcal{L} = 0$ and $\delta \mathrm{d}\mathcal{L} = 0$.

  Take $\Gamma = \partial B$, then $\delta \int_\Gamma \mathcal{L} = 0 = \delta \int_B \mathrm{d}\mathcal{L} = 0$

  [YB Suris, MV, 2016]

  but first put to good use by [D Sleigh, F Nijhoff, V Caudrelier, 2020].

- complementary form: $\delta \int_{\mathbb{R}^n} \mathcal{L} \wedge \varphi = 0$ for all exact $(n-d)$-forms φ.

  Vary u: $\delta \int_{\mathbb{R}^n} \mathcal{L} \wedge \mathrm{d}\alpha = \int_{\mathbb{R}^n} \delta\mathcal{L} \wedge \mathrm{d}\alpha = \pm \int_{\mathbb{R}^n} \delta \mathrm{d}\mathcal{L} \wedge \alpha = 0$

  Vary $\varphi = \mathrm{d}\alpha$: $\delta \int_{\mathbb{R}^n} \mathcal{L} \wedge \mathrm{d}\alpha = \pm \int_{\mathbb{R}^n} \mathcal{L} \wedge \mathrm{d}\delta\alpha = \pm \int_{\mathbb{R}^n} \mathrm{d}\mathcal{L} \wedge \delta\alpha = 0$

  Allow distributional forms (“currents”) to recover $\delta \int_\Gamma \mathcal{L} = 0$ as a special case.

  [Work in progress by D Vallenzasca, based on idea of R Izquierdo López]

Semi-discrete systems

Example: Toda lattice. One discrete dimension, many continuous times.

$$q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q}) \tag{T1}$$

$$q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q}), \tag{T2}$$

Semi-discrete Lagrangian 2-form with the coefficients (index 0: discrete direction)

$$\begin{aligned} L_{01} &= \tfrac{1}{2} q_1^2 - \exp(\bar{q} - q), \\ L_{02} &= q_1 q_2 - \tfrac{1}{3} q_1^3 - (q_1 + \bar{q}_1) \exp(\bar{q} - q), \\ L_{12} &= \tfrac{1}{4} \left(\bar{q}_2 - \bar{q}_{11} - \bar{q}_1^2 \right)^2 . \end{aligned}$$

The multiform Euler-Lagrange equations are (T1)–(T2) and

$$\tfrac{1}{2} q_{22} - q_{11} q_2 - 2 q_{12} q_1 - \tfrac{1}{2} q_{1111} + 3 q_1^2 q_{11} = 0.$$

The Lagrangian multiform produces a scalar PDE at a single lattice site.

[D Sleigh, MV, 2022]

Q Unified treatment of integrable hierarchies and their (semi-)discretisations?

Semi-discrete 3-form for KP proposed in [FW Nijhoff, 2024]

Bi-Hamiltonian systems

$$u_i = \mathcal{A}\frac{\delta H_i}{\delta u} = \mathcal{B}\frac{\delta H_{i-1}}{\delta u}, \ \mathcal{A} \text{ constant.}$$

Change to potential variable $\bar{u}$

Defined by $u = \mathcal{A}\bar{u}$. Then

- $\frac{\delta H}{\delta u} = -\mathcal{A}\frac{\delta H}{\delta \bar{u}}$
- $\mathcal{A}\bar{u}_i = -\frac{\delta H_i}{\delta \bar{u}}$ and $\mathcal{B}\bar{u}_i = -\frac{\delta H_{i+1}}{\delta \bar{u}}$
- $\mathcal{A}$ and $\mathcal{B}$ are symplectic operators in $\bar{u}$, so these are EL equations of

  $$L_{1i} = p[\bar{u}]\bar{u}_i - H_i[\bar{u}_x]$$

  and

  $$M_{1i} = q[\bar{u}]\bar{u}_i - H_{i+1}[\bar{u}_x]$$

$\rightsquigarrow$ two Lagrangian 2-forms $\mathcal{L}$, $\mathcal{M}$, giving evolutionary equations $\bar{u}_i = \cdots$.

Example, KdV

$$\mathcal{A} = \partial_x, \qquad \mathcal{B} = \partial_x^3 + 2u\partial_x + u_x,$$

$$H_1 = \tfrac{1}{2}u^2, \qquad H_2 = \tfrac{1}{2}u^3 - \tfrac{1}{2}u_x^2.$$

Lenard recursion

$$\begin{tikzcd}
\frac{\delta H_0}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & 0 & & \\
\frac{\delta H_1}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & u_{t_1} & \bar{u}_{t_1} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_1}{\delta \bar{u}} \\
\frac{\delta h_2}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & u_{t_2} & \bar{u}_{t_2} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_2}{\delta \bar{u}} \\
\frac{\delta h_3}{\delta u} \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & u_{t_3} & \bar{u}_{t_3} \arrow[l, "\mathcal{A}"'] \arrow[r, "\mathcal{A}"] \arrow[dr, "\mathcal{B}"] & -\frac{\delta h_3}{\delta \bar{u}} \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

Q What if neither $\mathcal{A}$ nor $\mathcal{B}$ is constant?

[P Vergallo, MV. 2026]

Gauged multiforms

Lie group G acting on T^*M, moment map $\mu : T^*M \to \mathfrak{g}^*$

Reduced phase space $\mu^{-1}(0)/G$.

Gauged Lagrangian 1-form

Introduce "Lagrange multiplier" $\mathcal{A}$, a $\mathfrak{g}$-valued 1-form on $\mathbb{R}^n$.

Action for univariational principle:

$$S_\Gamma[\Sigma, \mathcal{A}] = \int_0^1 \Gamma^* \left(\Sigma^* \mathcal{L} + \langle \sigma^* \mu, \mathcal{A} \rangle \right)$$

Gauge symmetry:

$$\begin{aligned} \Sigma &\mapsto g \cdot \Sigma \\ \mathcal{A} &\mapsto g \mathcal{A} g^{-1} - \mathrm{d}g g^{-1} \end{aligned}$$

Application: gauge theoretic origins of 1d integrable systems

3d BF theory ↔ Hitchin systems: Gaudin models, elliptic spin CM.

Q Lagrangian 2-forms and 4d Chern-Simons?

[V Caudrelier, D Harland, AA Singh, B Vicedo. 2026]

Quantisation

Path integral quantisation in Lagrangian multiform setting?

Two meanings of "path":

- Multiform path: curve Γ in multi-time (curve of integration in classical 1-form case)
- Feynman integral path: curve in phase space parametrised by time interval (single-time setting) or curve in multi-time (multiform setting), representing either the classical solution or a perturbation thereof.

Closure property $\mathrm{d}\mathcal{L} = 0$ implies that action is independent of Γ on classical solutions.

For harmonic oscillators, this is true away from classical solutions too.

[SD King, FW Nijhoff, 2019]

[T Kongkoom, S Yoo-Kong. 2023]

- Q How to account for choice of Γ for more general systems?
- Q Physical meaning of multi-time propagator?
- Q Quantise all directions on equal footing
  or approach as quantum system with classical symmetries?

Spectrality property

Parameter-dependent multiform $\mathcal{L}[u;\lambda]$ with double-zero property:

$$\mathrm{d}\mathcal{L}([u],\lambda) = A([u],\lambda)B([u],\lambda)$$

Then $\delta\mathrm{d}\mathcal{L} = 0$ on the system of equations $A = 0$, $B = 0$.

Double-zero property

Vary wrt $u \Rightarrow$ multiform EL equations follow from $A = 0$, $B = 0$.
(Often equivalent)

Spectrality property

Vary wrt $\lambda \Rightarrow$ conservation Law $\mathrm{d}\left(\frac{\partial \mathcal{L}}{\partial \lambda}\right) = 0$
partial derivatives of coefficients of $\mathcal{L}$ are conserved densities.

Formulated for discrete 1-forms in [R Boll, M Petrera, YB Suris. 2015]

Name from "spectrality" of Bäcklund transforms [VB Kuznetsov, EK Sklyanin. 1998]

Q Applications?

Special solutions and stability

Example: variational characterisation of KdV solitons an n-soliton profile is a minimiser of H_{n+1} constrained to a common level set of $H_1, \ldots, H_n$.

[P Lax. 1968, 1975]

[JH Maddocks, RL Sachs. 1993]

- Multiform formulation of such characterisation?
- When are multiform solutions minimisers?
  $\rightsquigarrow$ unique solution to boundary-value problem of EL equations.

From conservation law to multiform

Previously: $\begin{cases} \text{Lagrangian } L \\ \text{variational symmetry} \end{cases} \quad \Rightarrow \quad \text{multiform } \mathcal{L} \quad \text{with “}\mathrm{d}\mathcal{L} = AB\text{”}$

where $A = \dfrac{\delta L}{\delta u}$ and B is characteristic of symmetry.

What if we don't have a Lagrangian to start with?

Start from a conservation law for an equation $A = 0$, with characteristic B, i.e. $AB =$ divergence, without imposing that A is a variational derivative.

Example: Veronese Web hierarchy

$$A_{ijk} = (c^i - c^j)\frac{u_{ij}}{u_i u_j} + (c^j - c^k)\frac{u_{jk}}{u_j u_k} + (c^k - c^i)\frac{u_{ik}}{u_i u_k},$$

$$B_{ijk} = u_{ijk} - \frac{1}{2}\left(\frac{u_{ij}u_{ik}}{u_i} + \frac{u_{ij}u_{jk}}{u_j} + \frac{u_{ik}u_{jk}}{u_{ik}}\right),$$

then $\sum_{i,j,k} A_{ijk}B_{ijk}\,\mathrm{d}t^i \wedge \mathrm{d}t^j \wedge \mathrm{d}t^k = \mathrm{d}\mathcal{L}$, with

$$\mathcal{L} = \sum_{i<j}(c^i - c^j)\frac{u_{ij}^2}{u_i u_j}\,\mathrm{d}t^i \wedge \mathrm{d}t^j \qquad c^i = \text{const},$$

From conservation law to multiform

Recipe. Take an integrable PDE

$$A = 0$$

Consider a conservation law F, G, H such that

$$F_1 + G_2 + H_3 = A \cdot B.$$

Then the combined system

$$A = 0, \qquad B = 0$$

seems to be

- in involution (i.e. consistent/commuting)
- variational – it characterises stationary points of the Lagrangian multiform principle.

Result: a 2-form characterising a system of 3d PDEs.

[EV Ferapontov, MV. 2025]

Q Under what conditions does this hold?

Conclusion

Con~~clusion~~ tents

1. Prelude: the discrete case
2. Lagrangian 1-forms and integrable ODEs
3. Lagrangian multiforms for integrable PDEs
4. Construction of Lagrangian multiforms
5. Non-commutativity
6. Recent progress and open questions
   - Alternative formulations of the multiform variational principle
   - Semi-discrete systems
   - Bi-Hamiltonian systems
   - Gauged multiforms
   - Quantisation
   - Spectrality
   - Special solutions and stability
   - Multiforms from conservation laws

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Thank you for your attention

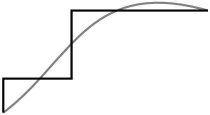
Derivation of the multiform Euler-Lagrange equations

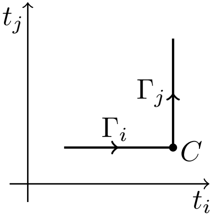
Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[q] \, \mathrm{d}t_i$, with

$$\begin{aligned} L_1[q] &= L_1(q, q_{t_1}), \\ L_i[q] &= L_i(q, q_{t_1}, q_{t_i}), \quad i \neq 1 \end{aligned}$$

Lemma

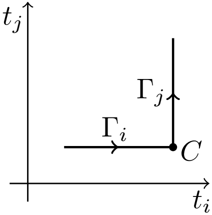
If the action $\int_\Gamma \mathcal{L}$ is critical on all stepped curves Γ in $\mathbb{R}^N$, then it is critical on all smooth curves.





Variations are local, so it is sufficient to look at one corner $\Gamma = \Gamma_i \cup \Gamma_j$ at a time.

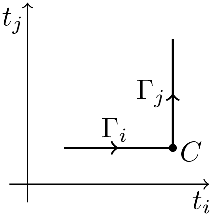
Derivation of the multiform Euler-Lagrange equations



On one of the straight pieces, Γ_i ($i \neq 1$), we get

$$\delta \int_{\Gamma_i} L_i \,\mathrm{d}t_i = \int_{\Gamma_i} \left(\frac{\partial L_i}{\partial q} \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} + \frac{\partial L_i}{\partial q_{t_i}} \delta q_{t_i} \right) \mathrm{d}t_i$$

Derivation of the multiform Euler-Lagrange equations



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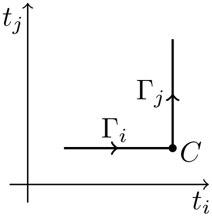
Integration by parts (wrt t_i only) yields

$$\delta \int_{\Gamma_i} L_i \, \mathrm{d}t_i = \int_{\Gamma_i} \left(\left(\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} \right) \delta q + \frac{\partial L_i}{\partial q_{t_1}} \delta q_{t_1} \right) \mathrm{d}t_i + \frac{\partial L_i}{\partial q_{t_i}} \delta q \bigg|_C$$

Since p is an interior point of the curve, we cannot set $\delta q(C) = 0$!

Derivation of the multiform Euler-Lagrange equations



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Since p is an interior point of the curve, we cannot set $\delta q(C) = 0$!

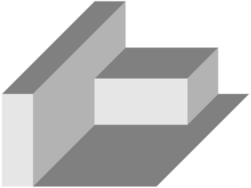
Arbitrary δq and δq_{t_1}, so we find:

Multiform Euler-Lagrange equations

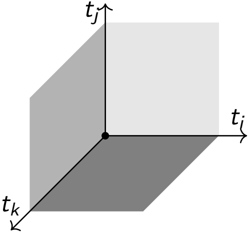
$$\frac{\partial L_i}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t_i} \frac{\partial L_i}{\partial q_{t_i}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_1}} = 0, \qquad \frac{\partial L_i}{\partial q_{t_i}} = \frac{\partial L_j}{\partial q_{t_j}}$$

Multiform EL equations

A smooth surface can be approximated to arbitrary precision by stepped surfaces.



It is sufficient to require criticality on stepped surfaces.



Variations can be taken locally, so it is sufficient to consider elementary corners.

Example: potential KdV hierarchy – weak multiform

The potential KdV equation $u_2 = u_{111} + 3u_1^2$ has a (weak) Lagrangian

$$L_{12} = \frac{1}{2} u_1 u_2 - \frac{1}{2} u_1 u_{111} - u_1^3 \qquad (u_k = u_{t_k})$$

and a hierarchy of variational symmetries, starting with

$$u_3 = Q[u] := u_{11111} + 10 u_1 u_{111} + 5 u_{11}^2 + 10 u_1^3.$$

This yields L_{13} and L_{23} such that

$$\frac{\mathrm{d} L_{23}}{\mathrm{d} t_1} - \frac{\mathrm{d} L_{13}}{\mathrm{d} t_2} + \frac{\mathrm{d} L_{12}}{\mathrm{d} t_3} = \frac{\delta L_{12}}{\delta u} (u_3 - Q)$$

Double zero property: multiform EL equations follow from

$$\frac{\delta L_{12}}{\delta u} = 0 \quad \Leftrightarrow \quad u_{12} = \frac{\mathrm{d}}{\mathrm{d} t_1} (u_{111} + 3u_1^2),$$
$$u_3 = u_{11111} + 10 u_1 u_{111} + 5 u_{11}^2 + 10 u_1^3.$$

Example: potential KdV hierarchy – weak multiform

$$L_{12} = \tfrac{1}{2} u_1 u_2 - \tfrac{1}{2} u_1 u_{111} - u_1^3$$

$$L_{13} = \tfrac{1}{2} u_1 u_3 - \tfrac{5}{2} u_1^4 - 5 u_1 u_{11}^2 - 5 u_1^2 u_{111} - \tfrac{1}{2} u_{111}^2$$

$$\begin{aligned} L_{23} = & -12 u_1^5 - 15 u_1^2 u_{11}^2 - 10 u_1^3 u_{111} + u_{11}^2 u_{111} - 2 u_1 u_{111}^2 - u_1 u_{11} u_{1111} - 5 u_1^2 u_{112} \\ & + 3 u_1^2 u_3 - \tfrac{1}{2} u_{1111}^2 - u_{111} u_{112} + \tfrac{1}{2} u_1 u_{113} + u_{1111} u_{12} - \tfrac{1}{2} u_{11} u_{13} + u_{111} u_3 \\ & - \tfrac{1}{2} u_2 u_3 \end{aligned}$$

Multiform Euler-Lagrange equations:

- The equations $\dfrac{\delta_{12} L_{12}}{\delta u} = 0$ and $\dfrac{\delta_{13} L_{13}}{\delta u} = 0$ yield

$$u_{12} = \frac{\mathrm{d}}{\mathrm{d} t_1} \left(u_{111} + 3 u_1^2 \right) \quad \text{and} \quad u_{13} = \frac{\mathrm{d}}{\mathrm{d} t_1} \left(u_{11111} + 10 u_1 u_{111} + 5 u_{11}^2 + 10 u_1^3 \right) .$$

- Equation $\dfrac{\delta_{13} L_{13}}{\delta u_1} = \dfrac{\delta_{23} L_{23}}{\delta u_2}$ yields

$$u_3 = u_{11111} + 10 u_1 u_{111} + 5 u_{11}^2 + 10 u_1^3 .$$

- The equations $\dfrac{\delta_{12} L_{12}}{\delta u_2} = \dfrac{\delta_{13} L_{13}}{\delta u_3}$ and $\dfrac{\delta_{12} L_{12}}{\delta u_1} = \dfrac{\delta_{32} L_{32}}{\delta u_3}$ are trivial

Example: potential KdV hierarchy – better multiform

Previous multiform satisfies

$$\begin{aligned}\frac{\mathrm{d}L_{23}}{\mathrm{d}t_1} - \frac{\mathrm{d}L_{13}}{\mathrm{d}t_2} + \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} &= \frac{\delta L_{12}}{\delta u}(u_3 - Q) \\ &= \frac{\mathrm{d}}{\mathrm{d}t_1}(-u_2 + u_{111} + 3u_1^2) \cdot (u_3 - Q)\end{aligned}$$

Replace L_{23} with

$$\widetilde{L}_{23} = L_{23} - \frac{1}{2}(-u_2 + u_{111} + 3u_1^2)(u_3 - Q),$$

then

$$\begin{aligned}\frac{\mathrm{d}\widetilde{L}_{23}}{\mathrm{d}t_1} - \frac{\mathrm{d}L_{13}}{\mathrm{d}t_2} + \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t_1}&(-u_2 + u_{111} + 3u_1^2) \cdot (u_3 - Q) \\ &- \frac{1}{2}(-u_2 + u_{111} + 3u_1^2) \cdot \frac{\mathrm{d}}{\mathrm{d}t_1}(u_3 - Q)\end{aligned}$$

Example: potential KdV hierarchy – better multiform

Previous multiform satisfies

$$\begin{aligned}\frac{\mathrm{d}L_{23}}{\mathrm{d}t_1} - \frac{\mathrm{d}L_{13}}{\mathrm{d}t_2} + \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} &= \frac{\delta L_{12}}{\delta u}(u_3 - Q) \\ &= \frac{\mathrm{d}}{\mathrm{d}t_1}(-u_2 + u_{111} + 3u_1^2) \cdot (u_3 - Q)\end{aligned}$$

Replace L_{23} with

$$\widetilde{L}_{23} = L_{23} - \frac{1}{2}(-u_2 + u_{111} + 3u_1^2)(u_3 - Q),$$

then

$$\begin{aligned}\frac{\mathrm{d}\widetilde{L}_{23}}{\mathrm{d}t_1} - \frac{\mathrm{d}L_{13}}{\mathrm{d}t_2} + \frac{\mathrm{d}L_{12}}{\mathrm{d}t_3} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t_1}(-u_2 + u_{111} + 3u_1^2) \cdot (u_3 - Q) \\ &\quad - \frac{1}{2}(-u_2 + u_{111} + 3u_1^2) \cdot \frac{\mathrm{d}}{\mathrm{d}t_1}(u_3 - Q)\end{aligned}$$

Double-zero property

If

$$\mathrm{d}\mathcal{L} = \sum_{i<j<k} (A^1_{ijk} B^1_{ijk} + \ldots + A^n_{ijk} B^n_{ijk})\, \mathrm{d}t^i \wedge \mathrm{d}t^j \wedge \mathrm{d}t^k,$$

then the system $A^\ell_{ijk} = 0$, $B^\ell_{ijk} = 0$ implies the multiform Euler-Lagrange equations

Example: potential KdV hierarchy – better multiform

Multiform Euler-Lagrange equations:

- The equations $\frac{\delta_{12} L_{12}}{\delta u} = 0$ and $\frac{\delta_{13} L_{13}}{\delta u} = 0$ yield
$$u_{12} = \frac{\mathrm{d}}{\mathrm{d}t_1}\left(u_{111} + 3u_1^2\right) \quad \text{and} \quad u_{13} = \frac{\mathrm{d}}{\mathrm{d}t_1}\left(u_{11111} + 10u_1 u_{111} + 5u_{11}^2 + 10u_1^3\right).$$
- Equation $\frac{\delta_{13} L_{13}}{\delta u_1} = \frac{\delta_{23} L_{23}}{\delta u_2}$ yields
$$u_3 = u_{11111} + 10u_1 u_{111} + 5u_{11}^2 + 10u_1^3.$$
- Equation $\frac{\delta_{12} L_{12}}{\delta u_1} = -\frac{\delta_{23} L_{23}}{\delta u_3}$ yields
$$u_2 = u_{111} + 3u_1^2.$$
- Equation $\frac{\delta_{12} L_{12}}{\delta u_2} = \frac{\delta_{13} L_{13}}{\delta u_3}$ is trivial.
- Equation $\frac{\delta_{12} L_{12}}{\delta u_{12}} - \frac{\delta_{13} L_{13}}{\delta u_{13}} + \frac{\delta_{23} L_{23}}{\delta u_{23}} = 0$ is trivial.