Symplectic operators and Lagrangian multiforms for bi-Hamiltonian systems

Mats Vermeeren

Joint work with Pierandrea Vergallo (University of Basilicata)

ISNMP Conference
Bad Ems

30 June 2026

Contents

Motivation

Lagrangian multiform theory (aka pluri-Lagrangian systems) [F Nijhoff & ..., 2009–...]

Variational principle for hierarchies of PDEs.

First step to construct: find a Lagrangian for one member of the hierarchy.

Example: KdV equation $u_t = u_{xxx} + 3uu_x$

- Traditional EL equations do not produce scalar evolutionary equations.
- Could try differentiating:

$$u_{xt} = u_{xxx} + 3uu_{xx} + 3u_x^2$$

  Still not an EL equation
- Need to pass to potential variable $\bar{u}$, defined by $u = \bar{u}_x$:

$$\bar{u}_{xt} = \bar{u}_{xxxx} + 3\bar{u}_x \bar{u}_{xx}$$

  EL equation of $\mathcal{L} = \frac{1}{2}\bar{u}_t \bar{u}_x - \frac{1}{2}\bar{u}_x \bar{u}_{xxx} - \bar{u}_x^3$.

Motivation

Why is $\bar{u}$ the right variable?

Is there a general way to turn a Hamiltonian PDE into an EL equation?

[Dorfman, 1987]

Dirac structures generalise both Hamiltonian and symplectic operators, which are key ingredients in the Hamiltonian resp. Lagrangian formulation.

[Mokhov, 1998]

Studies relations between invertible symplectic operators and Hamiltonian operators, considers classification problems, gives examples of bi-Lagrangian structures.

[Nutku & Pavlov, 2002]

Many examples of integrable PDEs with several non-equivalent Lagrangians, but the strategy to obtain them is not clearly explained.

[Bustamante & Hojman, 2003] [Pavlov & Vitolo, 2017] $\cdots$

Hamiltonian systems

In mechanics: $i_X \boldsymbol{\omega} = \mathrm{d}H$

Symplectic form ω defines an operator $\quad \Omega : TQ \to T^*Q$

$$\Omega X = \mathrm{d}H \qquad \text{(symplectic)}$$

ω nondegenerate, so Ω is invertible, $\quad A = \Omega^{-1} : T^*Q \to TQ$

$$X = A\mathrm{d}H \qquad \text{(Poisson)}$$

Hamiltonian systems

In mechanics: $i_X \boldsymbol{\omega} = \mathrm{d}H$

Symplectic form ω defines an operator $\quad \Omega : TQ \to T^*Q$

$$\Omega X = \mathrm{d}H \qquad \text{(symplectic)}$$

ω nondegenerate, so Ω is invertible, $\quad A = \Omega^{-1} : T^*Q \to TQ$

$$X = A\mathrm{d}H \qquad \text{(Poisson)}$$

In classical field theory: (Poisson) generalises to

$$u_t = \mathcal{A}\frac{\delta H}{\delta u}$$

$\mathcal{A}$ may not be invertible, so cannot write analogue of (symplectic).

Example: KdV can be written with $\mathcal{A} = \partial_x$ and $H = -\frac{1}{2}u_x^2 + \frac{1}{2}u^3$:

$$u_t = \partial_x(u_{xx} + \tfrac{3}{2}u^2)$$

Observation: Potential variable $\bar{u}$ defined through $\mathcal{A}$: $u = \bar{u}_x = \mathcal{A}u$

Lagrangian mechanics in phase space

Lagrangian for (symplectic)

Lagrangian:

$$\mathcal{L} = \frac{1}{2}\boldsymbol{z}^\top \Omega \boldsymbol{z}_t - H(\boldsymbol{z})$$

Euler-Lagrange equation:

$$\Omega \boldsymbol{z}_t = \nabla H(\boldsymbol{z})$$

In Darboux coordinates,

$$\mathcal{L} = \sum_i q^i p_i - H$$

$(\boldsymbol{z} : \mathbb{R} \to T^*Q)$

Lagrangian for (Poisson)

Introduce new variables $\bar{\boldsymbol{z}}$ by $\boldsymbol{z} = A\bar{\boldsymbol{z}}$

Lagrangian:

$$\mathcal{L} = \tfrac{1}{2}\bar{\boldsymbol{z}}^\top A \bar{\boldsymbol{z}}_t + H(\bar{\boldsymbol{z}})$$

Euler-Lagrange equation:

$$A\bar{\boldsymbol{z}}_t = -\nabla_{\bar{\boldsymbol{z}}}(H(A\bar{\boldsymbol{z}}))$$

$$\Downarrow A \text{ cnst}, \ \nabla_{\bar{\boldsymbol{z}}} = A^\top \nabla_{\boldsymbol{z}}$$

$$\boldsymbol{z}_t = -A^\top \nabla_{\boldsymbol{z}} H(\boldsymbol{z})$$

$$\Downarrow A \text{ skew-symmetric}$$

$$\boldsymbol{z}_t = A \nabla_{\boldsymbol{z}} H(\boldsymbol{z})$$

Does this generalise to Hamiltonian PDEs $u_t = \mathcal{A}\dfrac{\delta H}{\delta u}$?

Contents

Potential variable as dual space variable

- Space of independent space-variables $\mathbb{R}$.
- Dependent variables take values in vector space U.
- Dual space $\bar{U}$, bilinear pairing $(\cdot,\cdot) : U \times \bar{U} \to \mathbb{R}$.
- Phase space $\mathcal{F} = \{\mathbb{R} \to U \mid \text{smooth, rapidly decreasing}\}$,
  and smooth dual $\bar{\mathcal{F}} = \{\mathbb{R} \to \bar{U} \mid \text{smooth, rapidly decreasing}\}$.
- Pairing extends to $\langle \cdot,\cdot \rangle = \int (\cdot,\cdot) \mathrm{d}x : \mathcal{F} \times \bar{\mathcal{F}} \to \mathbb{R}$.

Potential variable as dual space variable

- Space of independent space-variables $\mathbb{R}$.
- Dependent variables take values in vector space U.
- Dual space $\bar{U}$, bilinear pairing $(\cdot,\cdot) : U \times \bar{U} \to \mathbb{R}$.
- Phase space $\mathcal{F} = \{\mathbb{R} \to U \mid \text{smooth, rapidly decreasing}\}$, and smooth dual $\bar{\mathcal{F}} = \{\mathbb{R} \to \bar{U} \mid \text{smooth, rapidly decreasing}\}$.
- Pairing extends to $\langle \cdot,\cdot \rangle = \int (\cdot,\cdot) \mathrm{d}x : \mathcal{F} \times \bar{\mathcal{F}} \to \mathbb{R}$.

Trivialised (co-)tangent bundles: $T\mathcal{F} = \mathcal{F} \times \mathcal{F}$ $\quad T\bar{\mathcal{F}} = \bar{\mathcal{F}} \times \bar{\mathcal{F}}$

$T^*\mathcal{F} = \mathcal{F} \times \bar{\mathcal{F}}$ $\quad T\bar{\mathcal{F}} = \bar{\mathcal{F}} \times \mathcal{F}$

Hamiltonian operator: $\mathcal{A} : \Gamma(T^*\mathcal{F}) \to \Gamma(T\mathcal{F})$, fibre-preserving, certain properties.

Constant Hamiltonian operator: given in every fibre by $A : \bar{\mathcal{F}} \to \mathcal{F}$.

Potential Hamiltonian variable $\bar{u}$, defined by $u = \mathcal{A}\bar{u}$

The phase space is now $\bar{\mathcal{F}}$. $\bar{U}$ becomes the primary space and U the dual.

Variational derivative as exterior derivative

Variational derivative $\frac{\delta F}{\delta u} \in \Gamma(T^*\mathcal{F})$ of a functional $F : \mathcal{F} \to \mathbb{R}$ defined by

$$\left\langle \frac{\delta F}{\delta u}, X \right\rangle = XF \qquad \text{for all vector fields } X \in \Gamma(T\mathcal{F})$$

If F has a density $F = \int f \mathrm{d}x$, this agrees with

$$\frac{\delta F}{\delta u} = \sum_{k \geq 0} (-1)^k \partial_x^k \frac{\partial f}{\partial u_{x^k}}$$

Variational derivative as exterior derivative

Variational derivative $\frac{\delta F}{\delta u} \in \Gamma(T^*\mathcal{F})$ of a functional $F : \mathcal{F} \to \mathbb{R}$ defined by

$$\left\langle \frac{\delta F}{\delta u}, X \right\rangle = XF \qquad \text{for all vector fields } X \in \Gamma(T\mathcal{F})$$

If F has a density $F = \int f \mathrm{d}x$, this agrees with

$$\frac{\delta F}{\delta u} = \sum_{k \geq 0} (-1)^k \partial_x^k \frac{\partial f}{\partial u_{x^k}}$$

Dual

Variational derivative $\frac{\delta \bar{F}}{\delta \bar{u}} \in \Gamma(T^*\bar{\mathcal{F}})$ of a functional $\bar{F} : \bar{\mathcal{F}} \to \mathbb{R}$ defined by

$$\left\langle \frac{\delta \bar{F}}{\delta \bar{u}}, \bar{X} \right\rangle = \bar{X}\bar{F} \qquad \text{for all vector fields } \bar{X} \in \Gamma(T\bar{\mathcal{F}})$$

If F has a density $\bar{F} = \int \bar{f} \mathrm{d}x$, this agrees with

$$\frac{\delta \bar{F}}{\delta \bar{u}} = \sum_{k \geq 0} (-1)^k \partial_x^k \frac{\partial \bar{f}}{\partial \bar{u}_{x^k}}$$

Hamiltonian and symplectic operators

$\mathcal{A} : T^*\mathcal{F} \to T\mathcal{F}$ is a Hamiltonian operator on $\mathcal{F}$ if

- $\mathcal{A}$ is skew-adjoint: $\langle f, \mathcal{A}g \rangle = -\langle \mathcal{A}f, g \rangle$,
- $\{F, G\}_\mathcal{A} := \langle \frac{\delta F}{\delta u}, \mathcal{A}\frac{\delta G}{\delta u} \rangle$ satisfies the Jacobi identity,

$\mathcal{J} : T\mathcal{F} \to T^*\mathcal{F}$ is a symplectic operator on $\mathcal{F}$ if

- $\mathcal{J}$ is skew-adjoint: $\langle X, \mathcal{J}Y \rangle = -\langle \mathcal{J}X, Y \rangle$,
- $\omega(X, Y) := \langle X, \mathcal{J}Y \rangle$ defines a closed 2-form,

The following are equivalent:

- $\mathcal{J}$ is symplectic
- there exists a $p[u] = p(u, u_x, u_{xx}, \dots)$ such that $\mathcal{J} = \ell_p^* - \ell_p$
- $\mathcal{J} u_t$ is the Euler-Lagrange expression of $\mathcal{L} = p[u]u_t$

In particular, $\mathcal{J} u_t = -\dfrac{\delta H}{\delta u}$ is an Euler-Lagrange equation if $\mathcal{J}$ is symplectic.

- $\mathcal{A}$ is a Hamiltonian operator if and only if $\mathcal{A}^{-1}$ is a symplectic operator
- Every constant skew-symmetric operator is Hamiltonian/symplectic

Express (Poisson) wrt potential Hamiltonian variable

Lemma

If $\mathcal{A} : T^*\mathcal{F} \to T\mathcal{F}$ given fibre-wise by $A : \bar{\mathcal{F}} \to \mathcal{F}$ is a constant Hamiltonian operator and $H : \mathcal{F} \to \mathbb{R}$, then

$$\frac{\delta H \circ A}{\delta \bar{u}} = A^* \frac{\delta H}{\delta u}$$

Express (Poisson) wrt potential Hamiltonian variable

Lemma

If $\mathcal{A} : T^*\mathcal{F} \to T\mathcal{F}$ given fibre-wise by $A : \bar{\mathcal{F}} \to \mathcal{F}$ is a constant Hamiltonian operator and $H : \mathcal{F} \to \mathbb{R}$, then

$$\frac{\delta H \circ A}{\delta \bar{u}} = A^* \frac{\delta H}{\delta u} = -A \frac{\delta H}{\delta u}$$

Express (Poisson) wrt potential Hamiltonian variable

Lemma

If $\mathcal{A} : T^*\mathcal{F} \to T\mathcal{F}$ given fibre-wise by $A : \bar{\mathcal{F}} \to \mathcal{F}$ is a constant Hamiltonian operator and $H : \mathcal{F} \to \mathbb{R}$, then

$$\frac{\delta H \circ A}{\delta \bar{u}} = A^* \frac{\delta H}{\delta u} = -A \frac{\delta H}{\delta u}$$

Set $u = A\bar{u}$ and use the Lemma:

$$u_t = A \frac{\delta H[u]}{\delta u} \qquad \Longrightarrow \qquad A\bar{u}_t = -\frac{\delta H[A\bar{u}]}{\delta \bar{u}}$$

Since A is a symplectic operator on $\bar{\mathcal{F}}$, this is the EL equation of

$$\mathcal{L} = p[\bar{u}]\bar{u}_t - H[A\bar{u}]$$

Switching to the potential Hamiltonian variable $\bar{u}$

- switches the roles of $\mathcal{F}$ and $\bar{\mathcal{F}}$,
- turns the Hamiltonian operator A into a symplectic operator,
- turns a Hamiltonian PDE into an EL equation.

Contents

Example: KdV

$u_t = u_{xxx} + 3uu_x$ can be written as

$$\begin{aligned} u_t = & \qquad\qquad \partial_x \frac{\delta}{\delta u}\left(-\tfrac{1}{2}u_x^2 + \tfrac{1}{2}u^3\right) \\ = & \; (\partial_x^3 + 2u\partial_x + u_x)\frac{\delta}{\delta u}\left(\tfrac{1}{2}u^2\right) \end{aligned}$$

Use $\mathcal{A} = \partial_x$ to define potential variable, $u = \bar{u}_x$.

Not just $\mathcal{A}$, but also $\mathcal{B} = \partial_x^3 + 2\bar{u}_x\partial_x + \bar{u}_x$ is symplectic in $\bar{u}$-variables.

$$\mathcal{A} = \ell_p - \ell_p^* \text{ with } p = \tfrac{1}{2}\bar{u}_x$$

$$\mathcal{B} = \ell_q - \ell_q^* \text{ with } q = \tfrac{1}{2}\left(\bar{u}_{xxx} + \bar{u}_x^2\right)$$

so we have two Lagrangians:

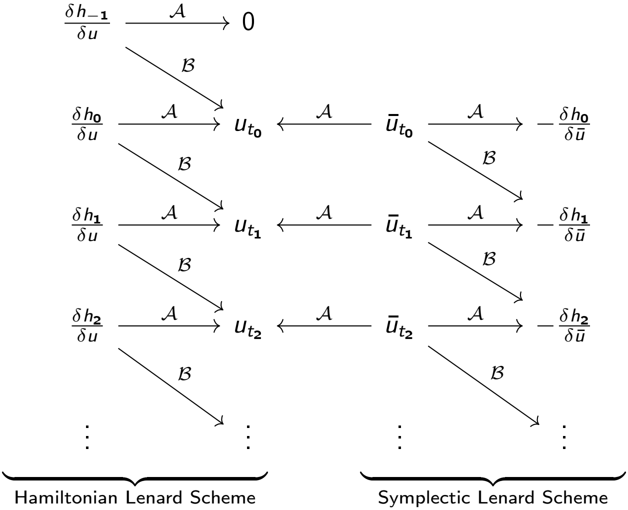
$$\begin{aligned} \mathcal{L}_\mathcal{A} &= \tfrac{1}{2}\bar{u}_x\bar{u}_t \qquad\qquad\qquad - \left(-\tfrac{1}{2}\bar{u}_{xx}^2 + \tfrac{1}{2}\bar{u}_x^3\right) \\ \mathcal{L}_\mathcal{B} &= \tfrac{1}{2}\left(\bar{u}_{xxx} + \bar{u}_x^2\right)\bar{u}_t - \left(\tfrac{5}{8}\bar{u}_x^4 - \tfrac{5}{2}\bar{u}_x\bar{u}_{xx}^2 + \tfrac{1}{2}\bar{u}_{xxx}^2\right) \end{aligned}$$

with EL equations

$$\mathcal{A}\left(-\bar{u}_t + \bar{u}_{xxx} + \tfrac{3}{2}\bar{u}_x^2\right) = 0$$

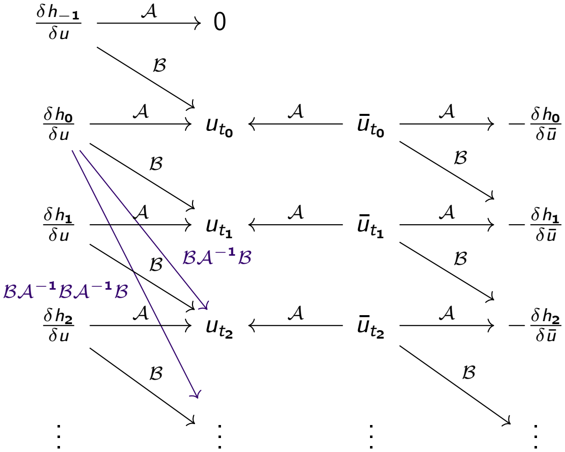
$$\mathcal{B}\left(-\bar{u}_t + \bar{u}_{xxx} + \tfrac{3}{2}\bar{u}_x^2\right) = 0$$

Double Lenard scheme



$\mathcal{A}, \mathcal{B}$ Hamiltonian pair, $\mathcal{A}$ constant, $\quad u = \mathcal{A}\bar{u}$, $\quad h_{-1}$ a Casimir of $\mathcal{A}$.

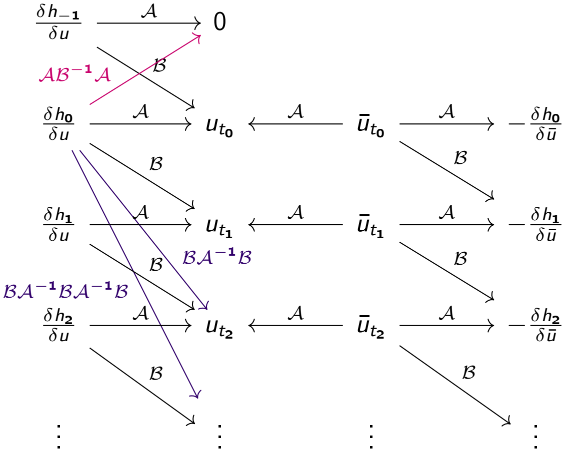
Hamiltonian recursion



Hierarchy of Hamiltonian operators obtained by recursion operator $\mathcal{B}\mathcal{A}^{-1}$:

$$\mathcal{A},\ \mathcal{B},\ \mathcal{B}\mathcal{A}^{-1}\mathcal{B},\ \mathcal{B}\mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1}\mathcal{B},\ \ldots$$

Hamiltonian recursion



Hierarchy of Hamiltonian operators obtained by recursion operator $\mathcal{B}\mathcal{A}^{-1}$:

$$\dots,\ \mathcal{A}\mathcal{B}^{-1}\mathcal{A}, \qquad \mathcal{A},\ \mathcal{B},\ \mathcal{B}\mathcal{A}^{-1}\mathcal{B},\ \mathcal{B}\mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1}\mathcal{B},\ \dots$$

Hierarchy of symplectic operators on $\bar{\mathcal{F}}$

Theorem

$\mathcal{A} : T^*\mathcal{F} \to T\mathcal{F}$ an invertible constant Hamiltonian operator, $A : \bar{\mathcal{F}} \to \mathcal{F}$
$\mathcal{B} : T^*\mathcal{F} \to T\mathcal{F}$ be a Hamiltonian operator compatible with $\mathcal{A}$

$\bar{\mathcal{A}} : T\bar{\mathcal{F}} \to T^*\bar{\mathcal{F}}$ constant symplectic operator defined by $A : \bar{\mathcal{F}} \to \mathcal{F}$
$\bar{\mathcal{B}} : T\bar{\mathcal{F}} \to T^*\bar{\mathcal{F}}$ defined fibre-wise by $\mathcal{B}[\bar{u}] : \bar{\mathcal{F}} \to \mathcal{F}$ as $\bar{\mathcal{B}}[\bar{u}] := \mathcal{B}[A\bar{u}]$

Then $\bar{\mathcal{A}}$, $\bar{\mathcal{B}}$ (and $\bar{\mathcal{A}}(\bar{\mathcal{A}}^{-1}\bar{\mathcal{B}})^n$ for $n \geq 0$) are symplectic operators on $\bar{\mathcal{F}}$.

Hierarchy of symplectic operators on $\bar{\mathcal{F}}$

Theorem

$\mathcal{A}: T^*\mathcal{F} \to T\mathcal{F}$ an invertible constant Hamiltonian operator, $A: \bar{\mathcal{F}} \to \mathcal{F}$
$\mathcal{B}: T^*\mathcal{F} \to T\mathcal{F}$ be a Hamiltonian operator compatible with $\mathcal{A}$

$\bar{\mathcal{A}}: T\bar{\mathcal{F}} \to T^*\bar{\mathcal{F}}$ constant symplectic operator defined by $A: \bar{\mathcal{F}} \to \mathcal{F}$
$\bar{\mathcal{B}}: T\bar{\mathcal{F}} \to T^*\bar{\mathcal{F}}$ defined fibre-wise by $\mathcal{B}[\bar{u}]: \bar{\mathcal{F}} \to \mathcal{F}$ as $\bar{\mathcal{B}}[\bar{u}] := \mathcal{B}[A\bar{u}]$

Then $\bar{\mathcal{A}}$, $\bar{\mathcal{B}}$ (and $\bar{\mathcal{A}}(\bar{\mathcal{A}}^{-1}\bar{\mathcal{B}})^n$ for $n \geq 0$) are symplectic operators on $\bar{\mathcal{F}}$.

Proof. $(\mathcal{A} + \lambda\mathcal{B}): T^*\mathcal{F} \to T\mathcal{F}$ is Hamiltonian for all λ, so

$$(\mathcal{A} + \lambda\mathcal{B})^{-1} = \mathcal{A}^{-1} - \lambda\mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1} + \lambda^2\mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1} - \dots$$

is symplectic on $\mathcal{F}$ for all $\lambda \quad \Rightarrow \quad (\mathcal{A}^{-1}\mathcal{B})^n\mathcal{A}^{-1}$ is symplectic on $\mathcal{F}$.

The 2-forms

$$\omega_n|_u(X, Y) = \left\langle X, (A^{-1}\mathcal{B}[u])^n A^{-1} Y \right\rangle$$
$$\bar{\omega}_n|_{\bar{u}}(\bar{X}, \bar{Y}) = \left\langle \bar{X}, A(A^{-1}\bar{\mathcal{B}}[\bar{u}])^n \bar{Y} \right\rangle$$

are related by pullback, $A^*\omega_n = -\bar{\omega}_n$. Hence $\mathrm{d}\omega_n = 0$ implies $\mathrm{d}\bar{\omega}_n = 0$. ■

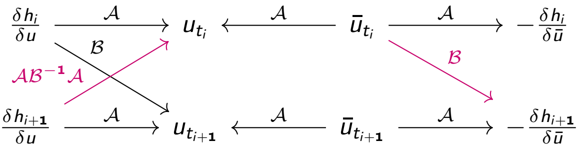
Dual picture

Hamiltonian

$$u_{t_{i+1}} = \mathcal{B}[u] \frac{\delta h_i}{\delta u}$$

Symplectic

$$\bar{\mathcal{B}}[\bar{u}]\, \bar{u}_{t_i} = \frac{\delta h_{i+1}}{\delta \bar{u}}$$



Recursion operator $\mathcal{B} \circ \mathcal{A}^{-1}$

Recursion operator $\mathcal{A}^{-1} \circ \mathcal{B}$

Corollary

The operators $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ are compatible symplectic operators.

Proof. The 2-forms from the previous proof are $\omega_0(X, Y) = \langle X, \bar{\mathcal{A}} Y \rangle$ and

$$\omega_k(X, Y) = \omega_0(X, \bar{\mathcal{R}}^k Y),$$

where $\bar{\mathcal{R}} = \bar{\mathcal{A}}^{-1} \bar{\mathcal{B}}$. It is a standard result [Dorfman, 1993] that $\bar{\mathcal{R}}$ must be a Nijenhuis operator if all ω_k are symplectic. ■

Contents

Variational principle for commuting flows

Suppose we have N commuting ODE flows with Lagrangians L_i. Consider

$$q : \mathbb{R}^N \to Q \quad \text{(multi-time to configuration space)}$$

Variational principle for commuting flows

Suppose we have N commuting ODE flows with Lagrangians L_i. Consider

$$q : \mathbb{R}^N \to Q \quad \text{(multi-time to configuration space)}$$

Pluri-Lagrangian principle [Yu Suris & ..., 2010–...]

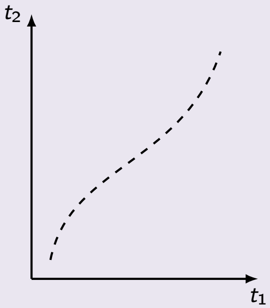
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q]\,\mathrm{d}t_i.$$

Look for $q(t_1, \ldots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$



Lagrangian multiform principle [F Nijhoff & ..., 2009–...]

Considers variations of the curve too: $\mathrm{d}\mathcal{L} = 0$ on solutions.

Variational principle for commuting flows

Suppose we have N commuting ODE flows with Lagrangians L_i. Consider

$$q : \mathbb{R}^N \to Q \quad \text{(multi-time to configuration space)}$$

Pluri-Lagrangian principle [Yu Suris & ..., 2010–...]

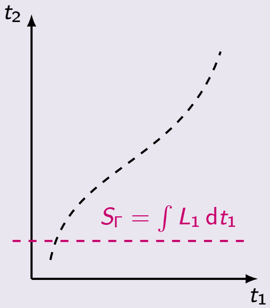
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^{N} L_i[q]\,\mathrm{d}t_i.$$

Look for $q(t_1, \ldots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[q]$$

is critical w.r.t. variations of q, simultaneously over every curve Γ in multi-time $\mathbb{R}^N$



Lagrangian multiform principle [F Nijhoff & ..., 2009–...]

Considers variations of the curve too: $\mathrm{d}\mathcal{L} = 0$ on solutions.

Multiform Euler-Lagrange equations

Pluri-Lagrangian principle is satisfied iff

$$\frac{\delta_i L_i}{\delta q_I} = 0 \qquad \forall I \not\ni t_i$$

$$\frac{\delta_i L_i}{\delta q_{It_i}} = \frac{\delta_j L_j}{\delta q_{It_j}} \qquad \forall I$$

where

- I denotes a multi-index (a combination of derivatives)
- $\dfrac{\delta_i}{\delta}$ denotes the variational derivative in the direction t_i:

$$\frac{\delta_i}{\delta q_I} = \sum_{\alpha \geq 0} (-1)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d} t_i^\alpha} \frac{\partial}{\partial q_{I t_i^\alpha}} \qquad = \frac{\partial}{\partial q_I} - \frac{\mathrm{d}}{\mathrm{d} t_i} \frac{\partial}{\partial q_{I t_i}} + \dots$$

Variational principle for PDEs: 2-forms (or higher forms)

Pluri-Lagrangian principle

Given a 2-form

$$\mathcal{L}[u] = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j,$$

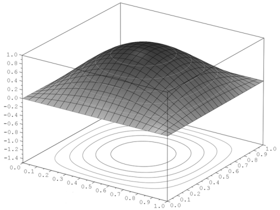
find a field $u(t_1, \dots t_N)$, such that

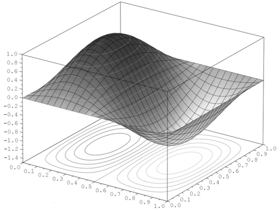
$$\int_\Gamma \mathcal{L}[u]$$

is critical on all smooth surfaces Γ in multi-time $\mathbb{R}^N$, w.r.t. variations of u.

Lagrangian multiform principle

Additionally require that $\mathrm{d}\mathcal{L}[u] = 0$ on solutions: action is critical w.r.t. deformations of the surface.





Multiform EL equations for a 2-form

Let $\mathcal{L}[u] = \sum_{i,j} L_{ij}[u] \, \mathrm{d}t_i \wedge \mathrm{d}t_j$.

Solutions to the pluri-Lagrangian principle are characterised by

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \qquad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{It_k}} \qquad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{It_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{It_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{It_k t_i}} = 0 \qquad \forall I,$$

where I denotes a multi-index (a combination of derivatives) and

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{\mathrm{d}^\alpha}{\mathrm{d}t_i^\alpha} \frac{\mathrm{d}^\beta}{\mathrm{d}t_j^\beta} \frac{\partial L_{ij}}{\partial u_{It_i^\alpha t_j^\beta}}$$

Exterior derivative

If the surface of integration is the boundary of a volume, $\Gamma = \partial B$, then

$$\int_\Gamma \mathcal{L} = \int_B \mathrm{d}\mathcal{L}.$$

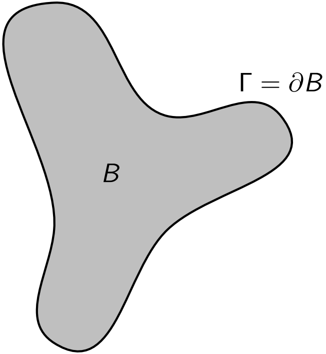
So a necessary (and in fact sufficient) condition for criticality is that infinitesimal variations of $\mathrm{d}\mathcal{L}$ vanish:

$$\delta \mathrm{d}\mathcal{L} = 0.$$

In other words:

$$\frac{\partial P_{ijk}}{\partial u_I} = 0 \qquad \forall I, \qquad\qquad (*)$$

where $\mathrm{d}\mathcal{L} = \sum_{i,j,k} P_{ijk}\, \mathrm{d}t^i \wedge \mathrm{d}t^i \wedge \mathrm{d}t^k$.



Exterior derivative

If the surface of integration is the boundary of a volume, $\Gamma = \partial B$, then

$$\int_\Gamma \mathcal{L} = \int_B \mathrm{d}\mathcal{L}.$$

So a necessary (and in fact sufficient) condition for criticality is that infinitesimal variations of $\mathrm{d}\mathcal{L}$ vanish:

$$\delta \mathrm{d}\mathcal{L} = 0.$$

In other words:

$$\frac{\partial P_{ijk}}{\partial u_I} = 0 \qquad \forall I, \qquad (*)$$

where $\mathrm{d}\mathcal{L} = \sum_{i,j,k} P_{ijk}\, \mathrm{d}t^i \wedge \mathrm{d}t^i \wedge \mathrm{d}t^k$.

Double-zero property

If the coefficients of $\mathrm{d}\mathcal{L}$ factorise,

$$\mathrm{d}\mathcal{L} = \sum_{i<j<k} A_{ijk} B_{ijk}\, \mathrm{d}t^i \wedge \mathrm{d}t^j \wedge \mathrm{d}t^k,$$

then on solutions of the system

$$A_{ijk} = 0, \qquad B_{ijk} = 0,$$

condition $(*)$ is satisfied:

$$\frac{\partial P_{ijk}}{\partial u_I} = \frac{\partial A_{ijk}}{\partial u_I} B_{ijk} + A_{ijk} \frac{\partial B_{ijk}}{\partial u_I} = 0$$

Hence the system $A_{ijk} = 0$, $B_{ijk} = 0$ implies the multiform Euler-Lagrange equations

Constructing a multiform for a given symplectic operator

Given a symplectic operator $\mathcal{A} = \ell_p - \ell_p^*$ and commuting Hamiltonians $h_1, h_2, \ldots$
Define

$$L_{0j} = p[\bar{u}]\bar{u}_{t_j} - h_j[\bar{u}] \qquad \text{where } [\bar{u}] = (\bar{u}, \bar{u}_x, \bar{u}_{xx}, \ldots)$$

Traditional EL equations: $$\frac{\delta_{0j} L_{0j}}{\delta \bar{u}} = 0 \qquad \Leftrightarrow \qquad \mathcal{A}\bar{u}_{t_j} = -\frac{\delta_{0j} h_j}{\delta \bar{u}}$$

Constructing a multiform for a given symplectic operator

Given a symplectic operator $\mathcal{A} = \ell_p - \ell_p^*$ and commuting Hamiltonians $h_1, h_2, \ldots$
Define

$$L_{0j} = p[\bar{u}]\bar{u}_{t_j} - h_j[\bar{u}] \qquad \text{where } [\bar{u}] = (\bar{u}, \bar{u}_x, \bar{u}_{xx}, \ldots)$$

Traditional EL equations:
$$\frac{\delta_{0j} L_{0j}}{\delta \bar{u}} = 0 \qquad \Leftrightarrow \qquad \mathcal{A}\bar{u}_{t_j} = -\frac{\delta_{0j} h_j}{\delta \bar{u}}$$

Assume $\frac{\delta h_j}{\delta \bar{u}} = -\mathcal{A} Q_j$, so these are consequences of evolutionary equations:

$$\bar{u}_{t_j} = Q_j[\bar{u}],$$

Then

$$\langle Q_j, \mathcal{A} Q_k \rangle = \left\langle \mathcal{A}^{-1} \frac{\delta h_j}{\delta \bar{u}}, \frac{\delta h_k}{\delta \bar{u}} \right\rangle = \left\langle \frac{\delta h_j}{\delta u}, \mathcal{A} \frac{\delta h_k}{\delta u} \right\rangle = -\{h_j, h_k\} = 0$$

so $Q_j \mathcal{A} Q_k = \frac{\mathrm{d}}{\mathrm{d}x}(\cdots)$

Aiming to have a double-zero expression for $\mathrm{d}\mathcal{L}$, we compute

$$\partial_{t_k} L_{0j} - \partial_{t_j} L_{0k} = \tfrac{1}{2}(\bar{u}_{t_j} - Q_j)\mathcal{A}(\bar{u}_{t_k} - Q_k) - \tfrac{1}{2}(\bar{u}_{t_k} - Q_k)\mathcal{A}(\bar{u}_{t_j} - Q_j) + \frac{\mathrm{d}}{\mathrm{d}x} \underbrace{(\cdots)}_{L_{jk}}$$

Multiform EL equations are evolutionary

Theorem

Assume the differential operator $\mathcal{A}$ has constant order. Then $\bar{u}$ satisfies the Lagrangian multiform principle for $\mathcal{L} = \sum_{i<j} L_{ij} \mathrm{d}t^i \wedge \mathrm{d}t^j$ if and only if

$$\bar{u}_{t_j} = Q_j[\bar{u}] \qquad \forall j$$

Proof. The coefficients

$$P_{0jk} = \partial_k L_{0j} - \partial_j L_{0k} + \partial_0 L_{jk} = \tfrac{1}{2}(\bar{u}_{t_j} - Q_j)\mathcal{A}(\bar{u}_{t_k} - Q_k) - \tfrac{1}{2}(\bar{u}_{t_k} - Q_k)\mathcal{A}(\bar{u}_{t_j} - Q_j)$$

of $\mathrm{d}\mathcal{L}$ have a double zero on $\bar{u}_{t_j} = Q_j[\bar{u}]$.

For the other coefficients of $\mathrm{d}\mathcal{L}$, we find

$$\partial_x P_{ijk} = \partial_i P_{0jk} + \partial_j P_{0ki} + \partial_k P_{0ij},$$

so $\partial_x P_{ijk}$ and hence P_{ijk} also have the double zero property.

So equations $\bar{u}_{t_j} = Q_j[\bar{u}]$ imply that $\bar{u}$ satisfies the multiform principle.

Other implication: depending on the differential order of $\mathcal{A}$, we can find an equation in the system of multiform EL equations that is equivalent to $\bar{u}_{t_j} = Q_j$. ■

KdV: first multiform

$L_{0j} = \frac{1}{2}\bar{u}_x \bar{u}_{t_j} - h_j[\bar{u}]$

$$L_{01} = \frac{1}{2}\bar{u}_x \bar{u}_{t_1} - \frac{1}{2}\bar{u}_x^3 + \frac{1}{2}\bar{u}_{xx}^2$$
$$L_{02} = \frac{1}{2}\bar{u}_x \bar{u}_{t_2} - \frac{5}{8}\bar{u}_x^4 + \frac{5}{2}\bar{u}_x \bar{u}_{xx}^2 - \frac{1}{2}\bar{u}_{xxx}^2$$

By construction,

$$\frac{\delta_{0j} L_{0j}}{\delta \bar{u}} = -\partial_x(\bar{u}_{t_j} - Q_j).$$

L_{ij} for $i, j > 0$ are not pretty but easily computed, for example

$$\begin{aligned} L_{12} = {} & \frac{3}{8}\bar{u}_x^5 - \frac{15}{8}\bar{u}_x^2 \bar{u}_{xx}^2 + \frac{5}{2}\bar{u}_x^3 \bar{u}_{xxx} - \frac{5}{4}\bar{u}_x^3 \bar{u}_{t_1} + \frac{7}{4}\bar{u}_{xx}^2 \bar{u}_{xxx} + \frac{3}{2}\bar{u}_x \bar{u}_{xxx}^2 - 3\bar{u}_x \bar{u}_{xx} \bar{u}_{xxxx} \\ & + \frac{3}{4}\bar{u}_x^2 \bar{u}_{xxxxx} + 5\bar{u}_x \bar{u}_{xx} \bar{u}_{xt_1} - \frac{5}{4}\bar{u}_{xx}^2 \bar{u}_{t_1} - \frac{5}{2}\bar{u}_x \bar{u}_{xxx} \bar{u}_{t_1} + \frac{3}{4}\bar{u}_x^2 \bar{u}_{t_2} - \frac{1}{2}\bar{u}_{xxxx}^2 \\ & + \frac{1}{2}\bar{u}_{xxx} \bar{u}_{xxxxx} - \bar{u}_{xxx} \bar{u}_{xxt_1} + \bar{u}_{xxxx} \bar{u}_{xt_1} - \bar{u}_{xx} \bar{u}_{xt_2} - \frac{1}{2}\bar{u}_{xxxxx} \bar{u}_{t_1} + \frac{1}{2}\bar{u}_{xxx} \bar{u}_{t_2} \end{aligned}$$

so

$$\frac{\delta_{01} L_{01}}{\delta \bar{u}_x} + \frac{\delta_{12} L_{12}}{\delta \bar{u}_{t_2}} = \left(\frac{1}{2}\bar{u}_{t_1} - \frac{3}{2}\bar{u}_x^2 - \bar{u}_{xxx}\right) + \left(\frac{3}{4}\bar{u}_x^2 + \frac{1}{2}\bar{u}_{xxx}\right) = \frac{1}{2}\left(\bar{u}_{t_1} - \frac{3}{2}\bar{u}_x^2 - \bar{u}_{xxx}\right)$$

KdV: second multiform

$L_{0j} = \frac{1}{2}\left(\bar{u}_x^2 + \bar{u}_{xxx}\right)\bar{u}_{t_j} - h_{j+1}$:

$$L_{01} = \frac{1}{2}\left(\bar{u}_x^2 + \bar{u}_{xxx}\right)\bar{u}_{t_1} - \frac{5}{8}\bar{u}_x^4 + \frac{5}{2}\bar{u}_x\bar{u}_{xx}^2 - \frac{1}{2}\bar{u}_{xxx}^2$$
$$L_{02} = \frac{1}{2}\left(\bar{u}_x^2 + \bar{u}_{xxx}\right)\bar{u}_{t_2} - \frac{7}{8}\bar{u}_x^5 + \frac{35}{4}\bar{u}_x^2\bar{u}_{xx}^2 - \frac{7}{2}\bar{u}_x\bar{u}_{xxx}^2 + \frac{1}{2}\bar{u}_{xxxx}^2$$

By construction,

$$\frac{\delta_{0j}L_{0j}}{\delta\bar{u}} = -\mathcal{B}(\bar{u}_{t_j} - Q_j)$$

L_{ij} for $i, j > 0$ are enormous but easily computed, for example

$$\begin{aligned}
L_{12} = {} & \frac{5}{8}\bar{u}_x^6 + \frac{5}{4}\bar{u}_x^3\bar{u}_{xx}^2 + \frac{55}{8}\bar{u}_x^4\bar{u}_{xxx} - \frac{15}{8}\bar{u}_x^4\bar{u}_{t_1} + \frac{5}{8}\bar{u}_{xx}^4 - \frac{25}{4}\bar{u}_x\bar{u}_{xx}^2\bar{u}_{xxx} + \frac{45}{4}\bar{u}_x^2\bar{u}_{xxx}^2 \\
& + 4\bar{u}_x^3\bar{u}_{xxxxx} + \frac{5}{4}\bar{u}_x^3\bar{u}_{xxt_1} + \frac{55}{4}\bar{u}_x^2\bar{u}_{xx}\bar{u}_{xt_1} - \frac{15}{2}\bar{u}_x\bar{u}_{xx}^2\bar{u}_{t_1} - \frac{35}{4}\bar{u}_x^2\bar{u}_{xxx}\bar{u}_{t_1} + \bar{u}_x^3\bar{u}_{t_2} \\
& + 5\bar{u}_{xxx}^3 + \frac{5}{2}\bar{u}_{xx}\bar{u}_{xxx}\bar{u}_{xxxx} - \frac{5}{2}\bar{u}_x\bar{u}_{xxxx}^2 + \frac{1}{4}\bar{u}_{xx}^2\bar{u}_{xxxxx} + \frac{5}{2}\bar{u}_x\bar{u}_{xxx}\bar{u}_{xxxxx} - \frac{3}{2}\bar{u}_x\bar{u}_{xx}\bar{u}_{xxxxxx} \\
& + \frac{3}{4}\bar{u}_x^2\bar{u}_{xxxxxxx} + \frac{5}{4}\bar{u}_{xx}^2\bar{u}_{xxt_1} - \frac{9}{2}\bar{u}_x\bar{u}_{xxx}\bar{u}_{xxt_1} - \frac{3}{4}\bar{u}_x^2\bar{u}_{xxt_2} + 2\bar{u}_{xx}\bar{u}_{xxx}\bar{u}_{xt_1} + \frac{9}{2}\bar{u}_x\bar{u}_{xxxx}\bar{u}_{xt_1} \\
& - \frac{7}{2}\bar{u}_x\bar{u}_{xx}\bar{u}_{xt_2} - \frac{11}{2}\bar{u}_{xxx}^2\bar{u}_{t_1} - \frac{13}{2}\bar{u}_{xx}\bar{u}_{xxxx}\bar{u}_{t_1} - \frac{7}{2}\bar{u}_x\bar{u}_{xxxxx}\bar{u}_{t_1} + \bar{u}_{xx}^2\bar{u}_{t_2} + \frac{5}{2}\bar{u}_x\bar{u}_{xxx}\bar{u}_{t_2} \\
& - \frac{1}{2}\bar{u}_{xxxx}\bar{u}_{xxxxxx} + \frac{1}{2}\bar{u}_{xxx}\bar{u}_{xxxxxxx} + \bar{u}_{xxxx}\bar{u}_{xxxt_1} - \frac{1}{2}\bar{u}_{xxxxx}\bar{u}_{xxt_1} \\
& + \frac{1}{2}\bar{u}_{xxx}\bar{u}_{xxt_2} + \frac{1}{2}\bar{u}_{xxxxxx}\bar{u}_{xt_1} - \frac{1}{2}\bar{u}_{xxxx}\bar{u}_{xt_2} - \frac{1}{2}\bar{u}_{xxxxxxx}\bar{u}_{t_1} + \frac{1}{2}\bar{u}_{xxxxx}\bar{u}_{t_2}
\end{aligned}$$

so

$$\frac{\delta_{01}L_{01}}{\delta\bar{u}_{xxx}} + \frac{\delta_{12}L_{12}}{\delta\bar{u}_{xxt_2}} = \left(\frac{1}{2}\bar{u}_{t_1} - \bar{u}_{xxx}\right) + \left(-\frac{3}{4}\bar{u}_x^2 + \frac{1}{2}\bar{u}_{xxx}\right) = \frac{1}{2}\left(\bar{u}_{t_1} - \frac{3}{2}\bar{u}_x^2 - \bar{u}_{xxx}\right)$$

Contents

Other examples

[Nutku & Pavlov. Multi-Lagrangians for integrable systems. 2002.]

Present bi-Lagrangians for

- KdV
- polytropic gas dynamics
- Kaup-Boussinesq
- NLS
- ...

These are the same Lagrangians as we find, but their presentation suggest some educated guesswork was involved.

We seem to have a general method.

Multi-Lagrangian $\neq$ Lagrangian multiform

Can we forget about the Hamiltonian side?

How to express the compatiblity of two Lagrangians / two multiforms / two symplectic operators?

Invertible symplectic operators $\mathcal{I}$ and $\mathcal{J}$ are compatible

- if $\mathcal{I}^{-1}$ and $\mathcal{J}^{-1}$ are compatible Hamiltonian operators
- if $\mathcal{I}^{-1} + \lambda \mathcal{J}^{-1}$ is Hamiltonian
- if
$$(\mathcal{I}^{-1} + \lambda \mathcal{J}^{-1})^{-1} = \mathcal{I} + \lambda \mathcal{I}\mathcal{J}^{-1}\mathcal{I} + \lambda^2 \mathcal{I}\mathcal{J}^{-1}\mathcal{I}\mathcal{J}^{-1}\mathcal{I} + \dots$$
is symplectic
- if $\mathcal{I}, \quad \mathcal{I}\mathcal{J}^{-1}\mathcal{I}, \quad \mathcal{I}\mathcal{J}^{-1}\mathcal{I}\mathcal{J}^{-1}\mathcal{I}, \dots$ are symplectic
- if $\mathcal{I}\mathcal{J}^{-1}$ is a Nijenhuis operator

Conclusions

- A "potential" variable should be defined by a Hamiltonian operator.

  It is in a space dual to the original variable.
- This allows us to transform Hamiltonian operators into symplectic operators and find Lagrangians
- In the traditional calculus of variations, the "higher Lagrangians" give increasingly weak differential consequences.

  In Lagrangian multiform theory, all give evolutionary equations.

References

- Vergallo & V.

  Duality of Hamiltonian and Lagrangian formulations for integrable systems

  arXiv:2604.19655
- Dorfman. Dirac structures of integrable evolution equations. 1987.
- Mokhov. Symplectic and Poisson structures on loop spaces of smooth manifolds, and integrable systems. 1998.
- Nutku & Pavlov. Multi-Lagrangians for integrable systems. 2002.
- Suris & V. On the Lagrangian structure of integrable hierarchies. 2016

Conclusions

- A "potential" variable should be defined by a Hamiltonian operator.
  It is in a space dual to the original variable.
- This allows us to transform Hamiltonian operators into symplectic operators and find Lagrangians
- In the traditional calculus of variations, the "higher Lagrangians" give increasingly weak differential consequences.
  In Lagrangian multiform theory, all give evolutionary equations.

References

- Vergallo & V.
  Duality of Hamiltonian and Lagrangian formulations for integrable systems
  arXiv:2604.19655
- Dorfman. Dirac structures of integrable evolution equations. 1987.
- Mokhov. Symplectic and Poisson structures on loop spaces of smooth manifolds, and integrable systems. 1998.
- Nutku & Pavlov. Multi-Lagrangians for integrable systems. 2002.
- Suris & V. On the Lagrangian structure of integrable hierarchies. 2016

Thank you for your attention!