

Symplectic operators and Lagrangian multiforms for bi-Hamiltonian systems

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Motivation

Lagrangian multiform theory (aka pluri-Lagrangian systems)

[F Nijhoff & ..., 2009–...]

Variational principle for hierarchies of PDEs.

First step to construct: find a Lagrangian for one member of the hierarchy.

Example: KdV equation $u_t = u_{xxx} + 3uu_x$

- ▶ Traditional EL equations do not produce scalar evolutionary equations.
- ▶ Could try differentiating:

$$u_{xt} = u_{xxxx} + 3uu_{xx} + 3u_x^2$$

Still not an EL equation

- ▶ Need to pass to potential variable $\bar{u}$, defined by $u = \bar{u}_x$:

$$\bar{u}_{xt} = \bar{u}_{xxxx} + 3\bar{u}_x \bar{u}_{xx}$$

EL equation of $\mathcal{L} = \frac{1}{2}\bar{u}_t \bar{u}_x - \frac{1}{2}\bar{u}_x \bar{u}_{xxx} - \bar{u}_x^3$.

Motivation

Why is $\bar{u}$ the right variable?

Is there a general way to turn a Hamiltonian PDE into an EL equation?

[Dorfman, 1987]

Dirac structures generalise both Hamiltonian and symplectic operators, which are key ingredients in the Hamiltonian resp. Lagrangian formulation.

[Mokhov, 1998]

Studies relations between invertible symplectic operators and Hamiltonian operators, considers classification problems, gives examples of bi-Lagrangian structures.

[Nutku & Pavlov, 2002]

Many examples of integrable PDEs with several non-equivalent Lagrangians, but the strategy to obtain them is not clearly explained.

[Bustamante & Hojman, 2003]

[Pavlov & Vitolo, 2017]

...

Hamiltonian systems

In mechanics: $i_X \omega = dH$

Symplectic form ω defines an operator $\Omega : TQ \rightarrow T^*Q$

$$\Omega X = dH \quad (\text{symplectic})$$

ω nondegenerate, so Ω is invertible, $A = \Omega^{-1} : T^*Q \rightarrow TQ$

$$X = AdH \quad (\text{Poisson})$$

In classical field theory: (Poisson) generalises to

$$u_t = \mathcal{A} \frac{\delta H}{\delta u}$$

$\mathcal{A}$ may not be invertible, so cannot write analogue of (symplectic).

Example: KdV can be written with $\mathcal{A} = \partial_x$ and $H = -\frac{1}{2}u_x^2 + \frac{1}{2}u^3$:

$$u_t = \partial_x(u_{xx} + \frac{3}{2}u^2)$$

Observation: Potential variable $\bar{u}$ defined through $\mathcal{A}$: $u = \bar{u}_x = \mathcal{A}u$

Lagrangian mechanics in phase space

Lagrangian for (symplectic)

Lagrangian:

$$\mathcal{L} = \frac{1}{2} \mathbf{z}^\top \Omega \mathbf{z}_t - H(\mathbf{z})$$

Euler-Lagrange equation:

$$\Omega \mathbf{z}_t = \nabla H(\mathbf{z})$$

In Darboux coordinates,

$$\mathcal{L} = \sum_i q^i p_i - H$$

$$(\mathbf{z} : \mathbb{R} \rightarrow T^*Q)$$

Lagrangian for (Poisson)

Introduce new variables $\bar{\mathbf{z}}$ by $\mathbf{z} = A\bar{\mathbf{z}}$

Lagrangian:

$$\mathcal{L} = \frac{1}{2} \bar{\mathbf{z}}^\top A \bar{\mathbf{z}}_t + H(\bar{\mathbf{z}})$$

Euler-Lagrange equation:

$$A \bar{\mathbf{z}}_t = -\nabla_{\bar{\mathbf{z}}} (H(A\bar{\mathbf{z}}))$$

$$\Downarrow A \text{ cnst, } \nabla_{\bar{\mathbf{z}}} = A^\top \nabla_{\mathbf{z}}$$

$$\mathbf{z}_t = -A^\top \nabla_{\mathbf{z}} H(\mathbf{z})$$

$$\Downarrow A \text{ skew-symmetric}$$

$$\mathbf{z}_t = A \nabla_{\mathbf{z}} H(\mathbf{z})$$

Does this generalise to Hamiltonian PDEs $u_t = \mathcal{A} \frac{\delta H}{\delta u}$?

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Potential variable as dual space variable

- ▶ Space of independent space-variables $\mathbb{R}$.
- ▶ Dependent variables take values in vector space U .
- ▶ Dual space $\bar{U}$, bilinear pairing $(\cdot, \cdot) : U \times \bar{U} \rightarrow \mathbb{R}$.
- ▶ Phase space $\mathcal{F} = \{\mathbb{R} \rightarrow U \mid \text{smooth, rapidly decreasing}\}$,
and smooth dual $\bar{\mathcal{F}} = \{\mathbb{R} \rightarrow \bar{U} \mid \text{smooth, bounded}\}$.
- ▶ Pairing extends to $\langle \cdot, \cdot \rangle = \int (\cdot, \cdot) dx : \mathcal{F} \times \bar{\mathcal{F}} \rightarrow \mathbb{R}$.

Trivialised (co-)tangent bundles:

$$\begin{array}{ll} T\mathcal{F} \simeq \mathcal{F} \times \mathcal{F} & T\bar{\mathcal{F}} \simeq \bar{\mathcal{F}} \times \bar{\mathcal{F}} \\ T^*\mathcal{F} \simeq \mathcal{F} \times \bar{\mathcal{F}} & T\bar{\mathcal{F}} \simeq \bar{\mathcal{F}} \times \mathcal{F} \end{array}$$

Hamiltonian operator: $\mathcal{A} : \Gamma(T^*\mathcal{F}) \rightarrow \Gamma(T\mathcal{F})$, fibre-preserving, certain properties.

Constant Hamiltonian operator: given in every fibre by $A : \bar{\mathcal{F}} \rightarrow \mathcal{F}$.

Potential Hamiltonian variable $\bar{u}$, defined by $u = A\bar{u}$

The phase space is now $\bar{\mathcal{F}}$. $\bar{U}$ becomes the primary space and U the dual.

Variational derivative as exterior derivative

Variational derivative $\frac{\delta F}{\delta u} \in \Gamma(T^*\mathcal{F})$ of a functional $F : \mathcal{F} \rightarrow \mathbb{R}$ defined by

$$\left\langle \frac{\delta F}{\delta u}, X \right\rangle = XF \quad \text{for all vector fields } X \in \Gamma(T\mathcal{F})$$

If F has a density $F = \int f dx$, this agrees with

$$\frac{\delta F}{\delta u} = \sum_{k \geq 0} (-1)^k \partial_x^k \frac{\partial f}{\partial u_{x^k}} du$$

Dual

Variational derivative $\frac{\delta \bar{F}}{\delta \bar{u}} \in \Gamma(T^*\bar{\mathcal{F}})$ of a functional $\bar{F} : \bar{\mathcal{F}} \rightarrow \mathbb{R}$ defined by

$$\left\langle \frac{\delta \bar{F}}{\delta \bar{u}}, \bar{X} \right\rangle = \bar{X}\bar{F} \quad \text{for all vector fields } \bar{X} \in \Gamma(T\bar{\mathcal{F}})$$

If F has a density $\bar{F} = \int \bar{f} dx$, this agrees with

$$\frac{\delta \bar{F}}{\delta \bar{u}} = \sum_{k \geq 0} (-1)^k \partial_x^k \frac{\partial \bar{f}}{\partial \bar{u}_{x^k}} d\bar{u}$$

Hamiltonian and symplectic operators

see e.g. [Mokhov, 1998]

$\mathcal{A} : T^*\mathcal{F} \rightarrow T\mathcal{F}$ is a **Hamiltonian operator** on $\mathcal{F}$ if

- ▶ $\mathcal{A}$ is skew-adjoint: $\langle f, \mathcal{A}g \rangle = -\langle \mathcal{A}f, g \rangle$,
- ▶ $\{F, G\}_{\mathcal{A}} := \langle \frac{\delta F}{\delta u}, \mathcal{A} \frac{\delta G}{\delta u} \rangle$ satisfies the Jacobi identity,

$\mathcal{J} : T\mathcal{F} \rightarrow T^*\mathcal{F}$ is a **symplectic operator** on $\mathcal{F}$ if

- ▶ $\mathcal{J}$ is skew-adjoint: $\langle X, \mathcal{J}Y \rangle = -\langle \mathcal{J}X, Y \rangle$,
- ▶ $\omega(X, Y) := \langle X, \mathcal{J}Y \rangle$ defines a closed 2-form,

The following are equivalent:

- ▶ $\mathcal{J}$ is symplectic
- ▶ there exists a $p[u] = p(u, u_x, u_{xx}, \dots)$ such that $\mathcal{J} = \ell_p^* - \ell_p$
- ▶ $\mathcal{J}u_t$ is the Euler-Lagrange expression of $\mathcal{L} = p[u]u_t$

In particular, $\mathcal{J}u_t = -\frac{\delta H}{\delta u}$ is an Euler-Lagrange equation if $\mathcal{J}$ is symplectic.

- ▶ $\mathcal{A}$ is a Hamiltonian operator if and only if $\mathcal{A}^{-1}$ is a symplectic operator
- ▶ Every constant skew-symmetric operator is Hamiltonian/symplectic

Express (Poisson) wrt potential Hamiltonian variable

Lemma

(∞ -dim analog of $\nabla_{\bar{z}} = A^T \nabla_z$)

If $\mathcal{A} : T^*\mathcal{F} \rightarrow T\mathcal{F}$ given fibre-wise by $A : \bar{\mathcal{F}} \rightarrow \mathcal{F}$ is a constant Hamiltonian operator and $H : \mathcal{F} \rightarrow \mathbb{R}$, then

$$\frac{\delta H \circ A}{\delta \bar{u}} = A^* \frac{\delta H}{\delta u} = -A \frac{\delta H}{\delta u}$$

Set $u = A\bar{u}$ and use the Lemma:

$$u_t = A \frac{\delta H[u]}{\delta u} \implies A\bar{u}_t = -\frac{\delta H[A\bar{u}]}{\delta \bar{u}}$$

Since A is a symplectic operator on $\bar{\mathcal{F}}$, this is the EL equation of

$$\mathcal{L} = p[\bar{u}]\bar{u}_t - H[A\bar{u}]$$

Switching to the potential Hamiltonian variable $\bar{u}$

- ▶ switches the roles of $\mathcal{F}$ and $\bar{\mathcal{F}}$,
- ▶ turns the Hamiltonian operator A into a symplectic operator,
- ▶ turns a Hamiltonian PDE into an EL equation.

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Example: KdV

$u_t = u_{xxx} + 3uu_x$ can be written as

$$\begin{aligned}u_t &= \partial_x \frac{\delta}{\delta u} \left(-\frac{1}{2} u_x^2 + \frac{1}{2} u^3 \right) \\&= (\partial_x^3 + 2u\partial_x + u_x) \frac{\delta}{\delta u} \left(\frac{1}{2} u^2 \right)\end{aligned}$$

Use $\mathcal{A} = \partial_x$ to define potential variable, $u = \bar{u}_x$.

Not just $\mathcal{A}$, but also $\mathcal{B} = \partial_x^3 + 2\bar{u}_x\partial_x + \bar{u}_x$ is symplectic in $\bar{u}$ -variables.

$$\begin{aligned}\mathcal{A} &= \ell_p - \ell_p^* \text{ with } p = \frac{1}{2} \bar{u}_x \\ \mathcal{B} &= \ell_q - \ell_q^* \text{ with } q = \frac{1}{2} (\bar{u}_{xxx} + \bar{u}_x^2)\end{aligned}$$

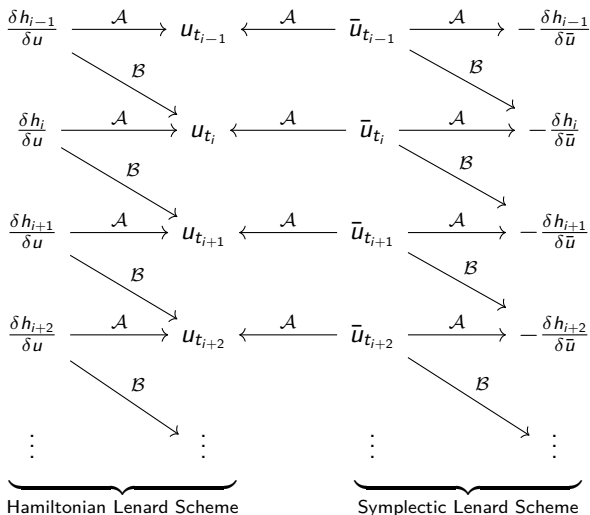
so we have two Lagrangians:

$$\begin{aligned}\mathcal{L}_{\mathcal{A}} &= \frac{1}{2} \bar{u}_x \bar{u}_t - \left(-\frac{1}{2} \bar{u}_{xx}^2 + \frac{1}{2} \bar{u}_x^3 \right) \\ \mathcal{L}_{\mathcal{B}} &= \frac{1}{2} (\bar{u}_{xxx} + \bar{u}_x^2) \bar{u}_t - \left(\frac{5}{8} \bar{u}_x^4 - \frac{5}{2} \bar{u}_x \bar{u}_{xx}^2 + \frac{1}{2} \bar{u}_{xxx}^2 \right)\end{aligned}$$

with EL equations

$$\begin{aligned}\mathcal{A}(-\bar{u}_t + \bar{u}_{xxx} + \frac{3}{2} \bar{u}_x^2) &= 0 \\ \mathcal{B}(-\bar{u}_t + \bar{u}_{xxx} + \frac{3}{2} \bar{u}_x^2) &= 0\end{aligned}$$

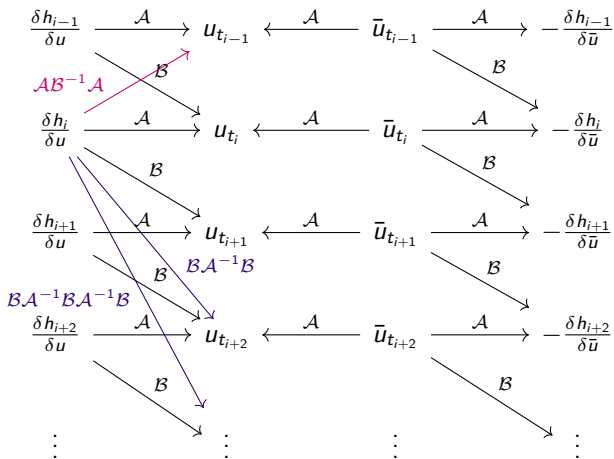
Double Lenard scheme



$\mathcal{A}, \mathcal{B}$ Hamiltonian pair, $\mathcal{A}$ constant,

$$u = \mathcal{A}\bar{u},$$

Hamiltonian recursion



Compatibility ensures that $\mathcal{A}^{-1}\mathcal{B}\frac{\delta h_i}{\delta u} \in \text{Im } \frac{\delta}{\delta u}$, so we can equate it to $\frac{\delta h_{i+1}}{\delta u}$.

Hierarchy of Hamiltonian operators obtained by recursion operator $\mathcal{B}\mathcal{A}^{-1}$:

$$\dots, \mathcal{A}\mathcal{B}^{-1}\mathcal{A}, \quad \mathcal{A}, \mathcal{B}, \mathcal{B}\mathcal{A}^{-1}\mathcal{B}, \mathcal{B}\mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1}\mathcal{B}, \dots$$

Hierarchy of symplectic operators on $\bar{\mathcal{F}}$

Theorem

$\mathcal{A} : T^*\mathcal{F} \rightarrow T\mathcal{F}$ an invertible constant Hamiltonian operator, $A : \bar{\mathcal{F}} \rightarrow \mathcal{F}$
 $\mathcal{B} : T^*\mathcal{F} \rightarrow T\mathcal{F}$ be a Hamiltonian operator compatible with $\mathcal{A}$

$\bar{\mathcal{A}} : T\bar{\mathcal{F}} \rightarrow T^*\bar{\mathcal{F}}$ constant symplectic operator defined by $A : \bar{\mathcal{F}} \rightarrow \mathcal{F}$

$\bar{\mathcal{B}} : T\bar{\mathcal{F}} \rightarrow T^*\bar{\mathcal{F}}$ defined fibre-wise by $\mathcal{B}[\bar{u}] : \bar{\mathcal{F}} \rightarrow \mathcal{F}$ as $\bar{\mathcal{B}}[\bar{u}] := \mathcal{B}[A\bar{u}]$

Then $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ (and $\bar{\mathcal{A}}(\bar{\mathcal{A}}^{-1}\bar{\mathcal{B}})^n$ for $n \geq 0$) are symplectic operators on $\bar{\mathcal{F}}$.

Proof. $(\mathcal{A} + \lambda\mathcal{B}) : T^*\mathcal{F} \rightarrow T\mathcal{F}$ is Hamiltonian for all λ , so

$$(\mathcal{A} + \lambda\mathcal{B})^{-1} = \mathcal{A}^{-1} - \lambda\mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1} + \lambda^2\mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1}\mathcal{B}\mathcal{A}^{-1} - \dots$$

is symplectic on $\mathcal{F}$ for all $\lambda \Rightarrow (\mathcal{A}^{-1}\mathcal{B})^n\mathcal{A}^{-1}$ is symplectic on $\mathcal{F}$.

The 2-forms

$$\omega_n|_u(X, Y) = \langle X, (\mathcal{A}^{-1}\mathcal{B}[u])^n\mathcal{A}^{-1}Y \rangle$$

$$\bar{\omega}_n|_{\bar{u}}(\bar{X}, \bar{Y}) = \langle \bar{X}, A(\mathcal{A}^{-1}\bar{\mathcal{B}}[\bar{u}])^n\bar{Y} \rangle$$

are related by pullback, $A^*\omega_n = -\bar{\omega}_n$. Hence $d\omega_n = 0$ implies $d\bar{\omega}_n = 0$. ■

Dual picture

Hamiltonian

$$u_{t_{i+1}} = \mathcal{B}[u] \frac{\delta h_i}{\delta u}$$

$$\begin{array}{ccc} \frac{\delta h_i}{\delta u} & \xrightarrow{\mathcal{A}} & u_{t_i} \\ & \searrow \mathcal{B} & \nearrow \mathcal{A} \\ \frac{\delta h_{i+1}}{\delta u} & \xrightarrow{\mathcal{A}} & u_{t_{i+1}} \end{array}$$

$\mathcal{A}\mathcal{B}^{-1}\mathcal{A}$

Recursion operator $\mathcal{B} \circ \mathcal{A}^{-1}$

Symplectic

$$\bar{\mathcal{B}}[\bar{u}] \bar{u}_{t_i} = \frac{\delta h_{i+1}}{\delta \bar{u}}$$

$$\begin{array}{ccc} \bar{u}_{t_i} & \xrightarrow{\mathcal{A}} & -\frac{\delta h_i}{\delta \bar{u}} \\ & \searrow \mathcal{B} & \nearrow \mathcal{A} \\ \bar{u}_{t_{i+1}} & \xrightarrow{\mathcal{A}} & -\frac{\delta h_{i+1}}{\delta \bar{u}} \end{array}$$

Recursion operator $\mathcal{A}^{-1} \circ \mathcal{B}$

Corollary

The operators $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ are compatible symplectic operators.

Proof. The 2-forms from the previous proof are $\omega_0(X, Y) = \langle X, \bar{\mathcal{A}}Y \rangle$ and

$$\omega_k(X, Y) = \omega_0(X, \bar{\mathcal{R}}^k Y),$$

where $\bar{\mathcal{R}} = \bar{\mathcal{A}}^{-1}\bar{\mathcal{B}}$. It is a standard result [Dorfman, 1993] that $\bar{\mathcal{R}}$ must be a Nijenhuis operator if all ω_k are symplectic. ■

Recap

- If $\mathcal{A}$ is a constant Hamiltonian operator for u , define $\bar{u}$ by $u = \bar{u}$, then $\mathcal{A}$ is a symplectic operator for $\bar{u}$:

$$u_t = \mathcal{A} \frac{\delta H}{\delta u} \qquad \mathcal{A} \bar{u}_t = \frac{\delta H}{\delta \bar{u}}$$

- If $\mathcal{B}$ is compatible with $\mathcal{A}$, then $\mathcal{B}$ is symplectic in the same $\bar{u}$ -variable::

$$u_{t_i} = \mathcal{B} \frac{\delta H_{i-1}}{\delta u} \qquad \mathcal{B} \bar{u}_{t_i} = \frac{\delta H_{i+1}}{\delta \bar{u}}$$

- Compatibility of Hamiltonian/symplectic operators is a necessary (and almost sufficient) condition for the Lenard iteration to work.
- A symplectic equation $\mathcal{A} \bar{u}_t = \frac{\delta H}{\delta \bar{u}}$ is an Euler-Lagrange equation.

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Variational principle for commuting flows

Suppose we have N commuting ODE flows with Lagrangians L_i . Consider

$$q : \mathbb{R}^N \rightarrow Q \quad (\text{multi-time to configuration space})$$

Pluri-Lagrangian principle

[Yu Suris & ..., 2010–...]

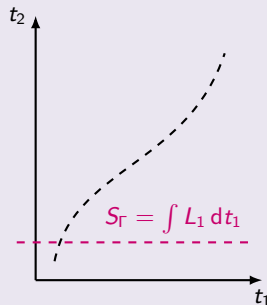
Combine the L_i into a 1-form

$$\mathcal{L}[q] = \sum_{i=1}^N L_i[q] dt_i.$$

Look for $q(t_1, \dots, t_N)$ such that the action

$$S_\Gamma = \int_\Gamma \mathcal{L}[q]$$

is critical w.r.t. variations of q , simultaneously over every curve Γ in multi-time $\mathbb{R}^N$



Lagrangian multiform principle

[F Nijhoff & ..., 2009–...]

Considers variations of the curve too: $d\mathcal{L} = 0$ on solutions.

Multiform Euler-Lagrange equations

Pluri-Lagrangian principle is satisfied iff

$$\begin{aligned}\frac{\delta_i L_i}{\delta q_I} &= 0 & \forall I \not\ni t_i \\ \frac{\delta_i L_i}{\delta q_{I t_i}} &= \frac{\delta_j L_j}{\delta q_{I t_j}} & \forall I\end{aligned}$$

where

- ▶ I denotes a multi-index (a combination of derivatives)
- ▶ $\frac{\delta_i}{\delta}$ denotes the variational derivative in the direction t_i :

$$\frac{\delta_i}{\delta q_I} = \sum_{\alpha \geq 0} (-1)^\alpha \frac{d^\alpha}{dt_i^\alpha} \frac{\partial}{\partial q_{I t_i^\alpha}} = \frac{\partial}{\partial q_I} - \frac{d}{dt_i} \frac{\partial}{\partial q_{I t_i}} + \dots$$

Example: central force problem

Take

$$L_1 = \frac{1}{2}|q_{t_1}|^2 - U(|q|)$$

$$L_2 = q_{t_1} \cdot q_{t_2} + (q_{t_1} \times q) \cdot \hat{v} \quad (\hat{v} \text{ fixed unit vector})$$

In general, for systems of Newtonian type,

$$L_i = q_{t_1} q_{t_i} - H_i(q, q_{t_1})$$

Multiform Euler-Lagrange equations of $\mathcal{L} = L_1 dt_1 + L_2 dt_2$

$$\frac{\partial L_1}{\partial q} - \frac{d}{dt_1} \frac{\partial L_1}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_1 t_1} = U'(|q|) \frac{q}{|q|} \quad (\text{central force})$$

$$\frac{\partial L_2}{\partial q} - \frac{d}{dt_2} \frac{\partial L_2}{\partial q_{t_2}} = 0 \quad \Rightarrow \quad q_{t_1 t_2} = \hat{v} \times q_{t_1}$$

$$\frac{\partial L_2}{\partial q_{t_1}} = 0 \quad \Rightarrow \quad q_{t_2} = \hat{v} \times q \quad (\text{rotation})$$

$$\frac{\partial L_1}{\partial q_{t_1}} = \frac{\partial L_2}{\partial q_{t_2}} \quad \Rightarrow \quad q_{t_1} = q_{t_1}$$

Variational principle for PDEs: 2-forms (or higher forms)

Pluri-Lagrangian principle

Given a 2-form

$$\mathcal{L}[u] = \sum_{i,j} L_{ij}[u] dt_i \wedge dt_j,$$

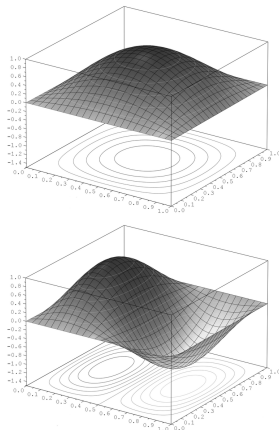
find a field $u(t_1, \dots, t_N)$, such that

$$\int_{\Gamma} \mathcal{L}[u]$$

is **critical on all smooth surfaces** Γ in multi-time $\mathbb{R}^N$, w.r.t. **variations of u** .

Lagrangian multiform principle

Additionally require that $d\mathcal{L}[u] = 0$ on solutions: action is critical w.r.t. deformations of the surface.



Multiform EL equations for a 2-form

Let $\mathcal{L}[u] = \sum_{i,j} L_{ij}[u] dt_i \wedge dt_j$.

Solutions to the pluri-Lagrangian principle are characterised by

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = 0 \quad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{I t_j}} = \frac{\delta_{ik} L_{ik}}{\delta u_{I t_k}} \quad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta u_{I t_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta u_{I t_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta u_{I t_k t_i}} = 0 \quad \forall I,$$

where I denotes a multi-index (a combination of derivatives) and

$$\frac{\delta_{ij} L_{ij}}{\delta u_I} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{d^\alpha}{dt_i^\alpha} \frac{d^\beta}{dt_j^\beta} \frac{\partial L_{ij}}{\partial u_{I t_i^\alpha t_j^\beta}}$$

Exterior derivative

If the surface of integration is the boundary of a volume, $\Gamma = \partial B$, then

$$\int_{\Gamma} \mathcal{L} = \int_B d\mathcal{L}.$$

So a necessary (and in fact sufficient) condition for criticality is that infinitesimal variations of $d\mathcal{L}$ vanish:

$$\delta d\mathcal{L} = 0.$$

In other words, if

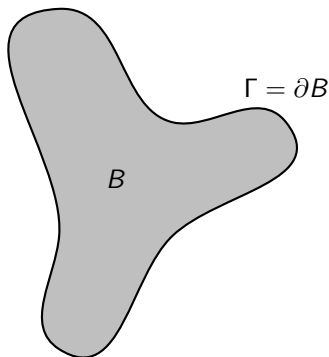
$$d\mathcal{L} = \sum_{i,j,k} P_{ijk} dt^i \wedge dt^j \wedge dt^k,$$

then

$$\frac{\partial P_{ijk}}{\partial t^l} = 0 \quad \forall l. \quad (*)$$

Double-zero property

If the coefficients of $d\mathcal{L}$ factorise,



Constructing a multiform for a given symplectic operator

Given a symplectic operator $\mathcal{A} = \ell_p - \ell_p^*$ and commuting Hamiltonians $h_1, h_2, \dots$

Define

$$L_{0j} = p[\bar{u}]\bar{u}_{t_j} - h_j[\bar{u}] \quad \text{where } [\bar{u}] = (\bar{u}, \bar{u}_x, \bar{u}_{xx}, \dots)$$

Traditional EL equations:
$$\frac{\delta_{0j} L_{0j}}{\delta \bar{u}} = 0 \quad \Leftrightarrow \quad \mathcal{A} \bar{u}_{t_j} = -\frac{\delta_{0j} h_j}{\delta \bar{u}}$$

Assume $\frac{\delta h_j}{\delta \bar{u}} = -\mathcal{A} Q_j$, so these are consequences of evolutionary equations:

$$\bar{u}_{t_j} = Q_j[\bar{u}],$$

Then

$$\langle Q_j, \mathcal{A} Q_k \rangle = \left\langle \mathcal{A}^{-1} \frac{\delta h_j}{\delta \bar{u}}, \frac{\delta h_k}{\delta \bar{u}} \right\rangle = \left\langle \frac{\delta h_j}{\delta \bar{u}}, \mathcal{A} \frac{\delta h_k}{\delta \bar{u}} \right\rangle = -\{h_j, h_k\} = 0$$

$$\text{so } Q_j \mathcal{A} Q_k = \frac{d}{dx}(\dots)$$

Aiming to have a double-zero expression for $d\mathcal{L}$, we compute

$$\partial_{t_k} L_{0j} - \partial_{t_j} L_{0k} = \frac{1}{2}(\bar{u}_{t_j} - Q_j) \mathcal{A}(\bar{u}_{t_k} - Q_k) - \frac{1}{2}(\bar{u}_{t_k} - Q_k) \mathcal{A}(\bar{u}_{t_j} - Q_j) + \frac{d}{dx} \underbrace{(\dots)}_{L_{jk}}$$

Multiform EL equations are evolutionary

Theorem

Assume the differential operator $\mathcal{A}$ has constant order. Then $\bar{u}$ satisfies the Lagrangian multiform principle for $\mathcal{L} = \sum_{i < j} L_{ij} dt^i \wedge dt^j$ if and only if

$$\bar{u}_{t_j} = Q_j[\bar{u}] \quad \forall j$$

Proof. The coefficients

$P_{0jk} = \partial_k L_{0j} - \partial_j L_{0k} + \partial_0 L_{jk} = \frac{1}{2}(\bar{u}_{t_j} - Q_j)\mathcal{A}(\bar{u}_{t_k} - Q_k) - \frac{1}{2}(\bar{u}_{t_k} - Q_k)\mathcal{A}(\bar{u}_{t_j} - Q_j)$
of $d\mathcal{L}$ have a double zero on $\bar{u}_{t_j} = Q_j[\bar{u}]$.

For the other coefficients of $d\mathcal{L}$, we find

$$\partial_x P_{ijk} = \partial_i P_{0jk} + \partial_j P_{0ki} + \partial_k P_{0ij},$$

so $\partial_x P_{ijk}$ and hence P_{ijk} also have the double zero property.

So equations $\bar{u}_{t_j} = Q_j[\bar{u}]$ imply that $\bar{u}$ satisfies the multiform principle.

Other implication: depending on the differential order of $\mathcal{A}$, we can find an equation in the system of multiform EL equations that is equivalent to $\bar{u}_{t_j} = Q_j$. ■

KdV: first multiform

$$L_{0j} = \frac{1}{2} \bar{u}_x \bar{u}_{t_j} - h_j[\bar{u}]$$

$$L_{01} = \frac{1}{2} \bar{u}_x \bar{u}_{t_1} - \frac{1}{2} \bar{u}_x^3 + \frac{1}{2} \bar{u}_{xx}^2$$

$$L_{02} = \frac{1}{2} \bar{u}_x \bar{u}_{t_2} - \frac{5}{8} \bar{u}_x^4 + \frac{5}{2} \bar{u}_x \bar{u}_{xx}^2 - \frac{1}{2} \bar{u}_{xxx}^2$$

By construction,

$$\frac{\delta_{0j} L_{0j}}{\delta \bar{u}} = -\partial_x (\bar{u}_{t_j} - Q_j).$$

L_{ij} for $i, j > 0$ are not pretty but easily computed, for example

$$\begin{aligned} L_{12} = & \frac{3}{8} \bar{u}_x^5 - \frac{15}{8} \bar{u}_x^2 \bar{u}_{xx}^2 + \frac{5}{2} \bar{u}_x^3 \bar{u}_{xxx} - \frac{5}{4} \bar{u}_x^3 \bar{u}_{t_1} + \frac{7}{4} \bar{u}_{xx}^2 \bar{u}_{xxx} + \frac{3}{2} \bar{u}_x \bar{u}_{xxx}^2 - 3 \bar{u}_x \bar{u}_{xx} \bar{u}_{xxxx} \\ & + \frac{3}{4} \bar{u}_x^2 \bar{u}_{xxxxx} + 5 \bar{u}_x \bar{u}_{xx} \bar{u}_{xt_1} - \frac{5}{4} \bar{u}_{xx}^2 \bar{u}_{t_1} - \frac{5}{2} \bar{u}_x \bar{u}_{xxx} \bar{u}_{t_1} + \frac{3}{4} \bar{u}_x^2 \bar{u}_{t_2} - \frac{1}{2} \bar{u}_{xxx}^2 \\ & + \frac{1}{2} \bar{u}_{xxx} \bar{u}_{xxxxx} - \bar{u}_{xxx} \bar{u}_{xxt_1} + \bar{u}_{xxxx} \bar{u}_{xt_1} - \bar{u}_{xx} \bar{u}_{xt_2} - \frac{1}{2} \bar{u}_{xxxxx} \bar{u}_{t_1} + \frac{1}{2} \bar{u}_{xxx} \bar{u}_{t_2} \end{aligned}$$

so

$$\frac{\delta_{01} L_{01}}{\delta \bar{u}_x} + \frac{\delta_{12} L_{12}}{\delta \bar{u}_{t_2}} = \left(\frac{1}{2} \bar{u}_{t_1} - \frac{3}{2} \bar{u}_x^2 - \bar{u}_{xxx} \right) + \left(\frac{3}{4} \bar{u}_x^2 + \frac{1}{2} \bar{u}_{xxx} \right) = \frac{1}{2} (\bar{u}_{t_1} - \frac{3}{2} \bar{u}_x^2 - \bar{u}_{xxx})$$

KdV: second multiform

$$L_{0j} = \frac{1}{2}(\bar{u}_x^2 + \bar{u}_{xxx})\bar{u}_{t_j} - h_{j+1}:$$

$$L_{01} = \frac{1}{2}(\bar{u}_x^2 + \bar{u}_{xxx})\bar{u}_{t_1} - \frac{5}{8}\bar{u}_x^4 + \frac{5}{2}\bar{u}_x\bar{u}_{xx}^2 - \frac{1}{2}\bar{u}_{xxx}^2$$

$$L_{02} = \frac{1}{2}(\bar{u}_x^2 + \bar{u}_{xxx})\bar{u}_{t_2} - \frac{7}{8}\bar{u}_x^5 + \frac{35}{4}\bar{u}_x^2\bar{u}_{xx}^2 - \frac{7}{2}\bar{u}_x\bar{u}_{xxx}^2 + \frac{1}{2}\bar{u}_{xxxx}^2$$

By construction,

$$\frac{\delta_{0j}L_{0j}}{\delta\bar{u}} = -\mathcal{B}(\bar{u}_{t_j} - Q_j)$$

L_{ij} for $i, j > 0$ are enormous but easily computed, for example

$$\begin{aligned} L_{12} = & \frac{5}{8}\bar{u}_x^6 + \frac{5}{4}\bar{u}_x^3\bar{u}_{xx}^2 + \frac{55}{8}\bar{u}_x^4\bar{u}_{xxx} - \frac{15}{8}\bar{u}_x^4\bar{u}_{t_1} + \frac{5}{8}\bar{u}_{xx}^4 - \frac{25}{4}\bar{u}_x\bar{u}_{xx}^2\bar{u}_{xxx} + \frac{45}{4}\bar{u}_x^2\bar{u}_{xxx}^2 \\ & + 4\bar{u}_x^3\bar{u}_{xxxx} + \frac{5}{4}\bar{u}_x^3\bar{u}_{xxt_1} + \frac{55}{4}\bar{u}_x^2\bar{u}_{xx}\bar{u}_{xt_1} - \frac{15}{2}\bar{u}_x\bar{u}_{xx}^2\bar{u}_{t_1} - \frac{35}{4}\bar{u}_x^2\bar{u}_{xxx}\bar{u}_{t_1} + \bar{u}_x^3\bar{u}_{t_2} \\ & + 5\bar{u}_{xxx}^3 + \frac{5}{2}\bar{u}_{xx}\bar{u}_{xxx}\bar{u}_{xxxx} - \frac{5}{2}\bar{u}_x\bar{u}_{xxx}^2 + \frac{1}{4}\bar{u}_{xx}^2\bar{u}_{xxxx} + \frac{5}{2}\bar{u}_x\bar{u}_{xxx}\bar{u}_{xxxx} - \frac{3}{2}\bar{u}_x\bar{u}_{xx}\bar{u}_{xxxxx} \\ & + \frac{3}{4}\bar{u}_x^2\bar{u}_{xxxxx} + \frac{5}{4}\bar{u}_x^2\bar{u}_{xxt_1} - \frac{9}{2}\bar{u}_x\bar{u}_{xxx}\bar{u}_{xt_1} - \frac{3}{4}\bar{u}_x^2\bar{u}_{xxt_2} + 2\bar{u}_{xx}\bar{u}_{xxx}\bar{u}_{xt_1} + \frac{9}{2}\bar{u}_x\bar{u}_{xxxx}\bar{u}_{xt_1} \\ & - \frac{7}{2}\bar{u}_x\bar{u}_{xx}\bar{u}_{xt_2} - \frac{11}{2}\bar{u}_{xxx}^2\bar{u}_{t_1} - \frac{13}{2}\bar{u}_{xx}\bar{u}_{xxx}\bar{u}_{t_1} - \frac{7}{2}\bar{u}_x\bar{u}_{xxxx}\bar{u}_{t_1} + \bar{u}_{xx}^2\bar{u}_{t_2} + \frac{5}{2}\bar{u}_x\bar{u}_{xxx}\bar{u}_{t_2} \\ & - \frac{1}{2}\bar{u}_{xxxx}\bar{u}_{xxxxx} + \frac{1}{2}\bar{u}_{xxx}\bar{u}_{xxxxx} + \bar{u}_{xxxx}\bar{u}_{xxt_1} - \frac{1}{2}\bar{u}_{xxxx}\bar{u}_{xt_1} \\ & + \frac{1}{2}\bar{u}_{xxx}\bar{u}_{xxt_2} + \frac{1}{2}\bar{u}_{xxxx}\bar{u}_{xt_1} - \frac{1}{2}\bar{u}_{xxxx}\bar{u}_{xt_2} - \frac{1}{2}\bar{u}_{xxxxx}\bar{u}_{t_1} + \frac{1}{2}\bar{u}_{xxxx}\bar{u}_{t_2} \end{aligned}$$

so

$$\frac{\delta_{01}L_{01}}{\delta\bar{u}_{xxx}} + \frac{\delta_{12}L_{12}}{\delta\bar{u}_{xxt_2}} = \left(\frac{1}{2}\bar{u}_{t_1} - \bar{u}_{xxx}\right) + \left(-\frac{3}{4}\bar{u}_x^2 + \frac{1}{2}\bar{u}_{xxx}\right) = \frac{1}{2}(\bar{u}_{t_1} - \frac{3}{2}\bar{u}_x^2 - \bar{u}_{xxx})$$

Contents

- 1 Introduction
- 2 Duality of Hamiltonian and symplectic operators
- 3 Bi-Hamiltonian systems
- 4 Lagrangian multiform theory
- 5 Discussion

Other examples

[Nutku & Pavlov. Multi-Lagrangians for integrable systems. 2002.]

Present bi-Lagrangians for

- ▶ KdV
- ▶ polytropic gas dynamics
- ▶ Kaup-Boussinesq
- ▶ NLS
- ▶ ...

These are the same Lagrangians as we find, but their presentation suggest some educated guesswork was involved.

We seem to have a general method.

Multi-Lagrangian $\neq$ Lagrangian multiform

Can we forget about the Hamiltonian side?

How to express the compatibility of two Lagrangians / two multiforms / two symplectic operators?

Invertible symplectic operators $\mathcal{I}$ and $\mathcal{J}$ are compatible

- ▶ if $\mathcal{I}^{-1}$ and $\mathcal{J}^{-1}$ are compatible Hamiltonian operators
- ▶ if $\mathcal{I}^{-1} + \lambda \mathcal{J}^{-1}$ is Hamiltonian
- ▶ if

$$(\mathcal{I}^{-1} + \lambda \mathcal{J}^{-1})^{-1} = \mathcal{I} + \lambda \mathcal{I} \mathcal{J}^{-1} \mathcal{I} + \lambda^2 \mathcal{I} \mathcal{J}^{-1} \mathcal{I} \mathcal{J}^{-1} \mathcal{I} + \dots$$

is symplectic

- ▶ if $\mathcal{I}$, $\mathcal{I} \mathcal{J}^{-1} \mathcal{I}$, $\mathcal{I} \mathcal{J}^{-1} \mathcal{I} \mathcal{J}^{-1} \mathcal{I}$, $\dots$ are symplectic
- ▶ if $\mathcal{I} \mathcal{J}^{-1}$ is a Nijenhuis operator

Conclusions

- ▶ A “potential” variable should be defined by a Hamiltonian operator.
It is in a space dual to the original variable.
- ▶ This allows us to transform Hamiltonian operators into symplectic operators and find Lagrangians
- ▶ In the traditional calculus of variations, the “higher Lagrangians” give increasingly weak differential consequences.
In Lagrangian multiform theory, all give evolutionary equations.

References

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- ▶ Nutku & Pavlov. Multi-Lagrangians for integrable systems. 2002.
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Thank you for your attention!